Immunizing Against Interest Rate Risk

1 Introduction

We are given a liability stream \( (L(t), t \geq 0) \) of obligations that must be met. We desire to purchase a portfolio of bonds of different types so as to roughly invest the present value of the liability stream, but do so in such a way that the portfolio is “immunized” against shifts in the term structure of interest rates. As interest rates change the prices of the bonds in the portfolio change as well as the present value of the liability stream. If we can basically ensure that for small perturbations in the term structure both the assets and the liabilities change in the same way, then in this sense we are immunized against interest rate risk.

2 Model Setup

Given a stream of cash flows \( (C_k(t), t \geq 0) \) associated with the \( k^{th} \) bond, its value \( P_k \) at time 0 is given by

\[
P_k = \int_0^\infty C_k(t) e^{-r(t)} dt,
\]

where the term structure of interest rates was previously estimated as \( (r(t), t \geq 0) \).

Let \( \delta(t) \) denote a perturbation of the term structure so that “tomorrow” the new term structure is given by \( r(t) + \delta(t) \). We assume that \( \delta(t) \) satisfies the following polynomial form:

\[
\delta(t) = \sum_{j=0}^J \alpha_j t^j.
\]

Let \( \alpha \) denote the vector of coefficients and define

\[
P_k(\alpha) := \int_0^\infty C_k(t) e^{-[r(t)+\sum_{j=0}^J \alpha_j t^j]} dt,
\]

which would be the new value of the \( k^{th} \) bond. Note that \( P_k(0) \) equals the current price of the bond, and we are, of course, assuming an instantaneous change. It is important to remember that even without interest rate changes, the bond price would increase as the time to maturity decreases. A portfolio consisting of \( n_k \) units of bond \( k \) would have its value changed to

\[
P(\alpha) := \sum_k n_k P_k(\alpha).
\]

We permit short selling, so the value of \( n_k \) could be negative. Analogously, we let \( L(\alpha) \) denote the new present value of the liability stream. Note that \( P(0) \) equals the current price of the

\[1\]See the Term Structure Estimation handout.
portfolio, and that \( L(0) \) equals the present value of the liability stream today.

Taking derivatives with respect to the components of \( \alpha \),

\[
\frac{\partial P_k(\alpha)}{\partial \alpha_j}|_{\alpha=0} = \int_0^\infty C_k(t) t^j e^{-r(t)t} dt := \Delta_{kj},
\]

(4)

and

\[
\frac{\partial P}{\partial \alpha_j}|_{\alpha=0} = \sum_k n_k \frac{\partial P_k(\alpha)}{\partial \alpha_j}|_{\alpha=0} = \sum_k n_k \Delta_{kj}.
\]

(5)

Analogously,

\[
\frac{\partial L}{\partial \alpha_j}|_{\alpha=0} = \int_0^\infty L(t) t^j e^{-r(t)t} dt := \Delta_{Lj}.
\]

(6)

Since

\[
P(\alpha) \approx P(0) + \sum_j \frac{\partial P}{\partial \alpha_j} \alpha_j,
\]

(7)

\[
L(\alpha) \approx L(0) + \sum_j \frac{\partial L}{\partial \alpha_j} \alpha_j,
\]

(8)

if we insist that

\[
P(0) = \sum_k n_k P_k = L(0),
\]

(9)

and that

\[
\frac{\partial P}{\partial \alpha_j} = \sum_k n_k \Delta_{kj} = \frac{\partial L}{\partial \alpha_j} = \Delta_{Lj},
\]

(10)

then to a first-order approximation we will immunize our portfolio against interest rate risk while satisfying our liability stream obligations. The immunization problem reduces to finding the values of the \( n_k \)’s to satisfy constraints (9) and (10). As these constraints are linear in the decision variables, our problem is to find a feasible solution to a linear program, with the usual caveat of replacing the variable \( n_k \) with \( n_k^+ - n_k^- \).

Of course, one should add limits on the amount of short selling, for example, by bounding the values of \( n_k^- \). One could also add sector constraints, for example, insisting that a certain percentage of the value of the initial portfolio be invested in a certain class of bonds.

### 3 Extensions

With so many bonds to choose from, and with each bond possessing different characteristics with respect to their partial derivatives, it may be beneficial to investigate relaxing the requirement (10) that the first-order derivatives of the portfolio match those of the liability stream.
By allowing a certain “tolerance” we may be able to acquire a portfolio at a reduced cost in relation to the present value of the liability stream.

Accordingly, one could choose to minimize the cost of the portfolio $\sum_k n_k P_k$, while insisting only that

$$| \sum_k n_k \Delta^j_{kj} - \Delta^j_{Lj} | \leq \epsilon,$$

(11)

where $\epsilon > 0$ denotes the level of tolerance. Of course, the constant $\epsilon$ could be replaced with a vector so that one could insist on different tolerances. Each partial derivative measures a different type of change in the ‘shape’ of the term structure, so one may have some a priori belief as to what type of change is more likely. One could trace out the efficient frontier, namely, the tradeoff between increasing $\epsilon$ and decreasing the cost of the portfolio.

It is also possible to model several possible scenarios for the liability stream, and then seek to immunize the portfolio in this broader sense. For the feasibility problem one could insist that the portfolio value track the expected value of the liability stream, or ensure that it covers it in all cases.

We close by noting that the term structure is used to fit a binomial lattice to represent the stochastic evolution of the short rates. Since the term structure is determined from the short rates, a change in the short term rate changes the term structure (according to the fit). Thus, it is conceptually possible to explore how the term structure changes as a function of the stochastic evolution of the short rate, and thus determine how the original portfolio is matching the obligations of the liability stream.