1. Let $v_1, v_2, \ldots, v_n$ be positive numbers. The *arithmetic mean* and *geometric mean* of these numbers are, respectively,

$$v_A := \frac{\sum_{i=1}^{n} v_i}{n} \quad \text{and} \quad v_G := \left( \prod_{i=1}^{n} v_i \right)^{1/n}.$$

(a) It is always true that $v_A \geq v_G$. Prove this inequality. (Hint: Use the fact that the $\ln(\cdot)$ function is concave.)

(b) If $r_1, r_2, \ldots, r_n$ are rates of return of a stock in each of $n$ periods, the arithmetic and geometric mean rates of return are, respectively,

$$r_A := \frac{\sum_{i=1}^{n} r_i}{n} \quad \text{and} \quad r_G := \left( \prod_{i=1}^{n} (1 + r_i) \right)^{1/n} - 1.$$

i. Suppose 40 is invested in the stock. During the first year the investment increases to 60 and during the second year it decreases to 48. Determine the arithmetic and geometric mean rates of return over the two years.

ii. Suppose 100 is invested in the stock. During the first year the investment increases to 150 and during the second year it decreases to 100. Determine the arithmetic and geometric mean rates of return over the two years.

(c) Which mean is more appropriate to describe investment performance?

2. A coin is flipped: if it comes up heads you win 0.25 and if it comes up tails you lose 0.25. The probability of the coin coming up heads is 0.60.

(a) Determine the mean and variance of your earnings at the end of the game if you play the game 10, 100 and 1 million times.

(b) Determine the probability your earnings will exceed 49,000 if you play the game 1 million times.

3. A stock price is governed by Geometric Brownian Motion with $\mu = 0.20$ and $\sigma = 0.40$. The initial price $S(0) = 1$. Evaluate the following quantities:

(a) $E[\ln S(1)]$ and the standard deviation of $\ln S(1)$.

(b) $E[S(1)]$ and the standard deviation of $S(1)$.

(c) The probability that $S(0.25) \leq 1.40$.

(d) The probability that the maximum value of the stock price over the next 3 months will not exceed 1.40.

(e) The probability that it will take longer than 1 year for the stock price to double in value.
(f) The probability an investor will make money on the following strategy: purchase 1 unit of the stock at time 0 and sell the stock at the first point in time at which either the price doubles or loses 50% of its initial value.

4. In the “Exotic Option” handout we used the binomial lattice to value a 5-month call option on a stock with a current price of 62, log-volatility of 20% per year, strike price of 60 and a risk-free rate of 10%. The call option value using this approach is 5.85. Using $S = 62$, $K = 60$, $\sigma = 0.20$ and $r = 0.10$, determine:

(a) the values of $d_1$ and $d_2$ in the Black-Scholes call option formula.

(b) the values for $\Phi(d_1)$ and $\Phi(d_2)$ using the following approximation (to within about six decimal places) $\Phi^a(\cdot)$ to $\Phi(\cdot)$ given by the modified polynomial relation

$$\Phi^a(x) := \begin{cases} 
1 - \phi(x) \left( a_1 k(x) + a - 2k(x)^2 + a_3 k(x)^3 + a_4 k(x)^4 + a_5 k(x)^5 \right) & \text{if } x \geq 0 \\
1 - \Phi^a(-x) & \text{if } x < 0,
\end{cases}$$

where $k(x) := (1 + \gamma x)^{-1}$, $\gamma = 0.2316419$, $a_1 = 0.319381530$, $a_2 = -0.35653782$, $a_3 = 1.781477937$, $a_4 = -1.821255978$, $a_5 = 1.330274429$.

(c) the Black-Scholes value for the call option using your answers to (a) and (b).

5. The current value of a stock is $S(0) = 100$. It has a continuously-compounded growth rate of $\nu = 12\%$ and a log-volatility $\sigma = 20\%$. Find appropriate parameters for the binomial lattice when the period length is 3 months. Draw the lattice and enter the node values for 1 year. Determine the probabilities of attaining the various final nodes.

6. Consider a call option with $S = 43$, $K = 40$, $\sigma = 0.20$, $r = 0.10$ and the time to expiration of 0.5 years.

(a) Determine the values for the call option $C$ and the greeks, $\Delta$, $\Gamma$ and $\Theta$.

(b) Suppose that in two weeks the stock price increases to 44.

i. Use the greeks to estimate the new value for the call option. Use Ito-Doeblin’s formula

$$dC \approx \frac{\partial C}{\partial S}(\Delta S) + \frac{1}{2} \frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + \frac{\partial C}{\partial t}(\Delta t).$$

ii. Determine the actual value of the call option.