Asset Price Dynamics Homework Solutions

1. (a) In general,
\[ \ln v_G = \sum_{i=1}^{n} \left( \frac{1}{n} \ln v_i \right) \leq \ln \left( \sum_{i=1}^{n} \frac{v_i}{n} \right), \quad \text{since } \ln(\cdot) \text{ is a concave function} \quad (1) \]
\[ = \ln v_A, \quad (2) \]
which implies \( v_G \leq v_A \), as required.

(b) i. \( r_1 = 0.50. \ r_2 = -0.20. \ v_A = 1/2(r_1 + r_2) = 0.15. \ v_G = \sqrt{(1 + r_1)(1 + r_2)} - 1 = 0.0954. \)

ii. \( r_1 = 0.50. \ r_2 = -0.33. \ v_A = 1/2(r_1 + r_2) = 0.083. \ v_G = \sqrt{(1 + r_1)(1 + r_2)} - 1 = 0. \)

(c) Usually the geometric mean rate of return is the most appropriate for measurement of investment performance.

2. (a) Using equations (38) and (41) of the notes we first calculate
\[ \sigma(2p - 1) = (0.25)(0.20) = 0.05; \ 4\sigma^2p(1 - p) = 0.06. \quad (3) \]
When \( n = 10 \) the mean is \( 10(0.05) = 0.5 \) and the variance is \( 10(0.06) = 0.6 \). When \( n = 100 \) the mean is \( 100(0.05) = 5 \) and the variance is \( 100(0.06) = 6 \). When \( n = 1 \) million the mean is \( 50,000 \) and the variance is \( 60,000 \).

(b) We seek
\[ P\{S_{1,000,000} \geq 49,000\} = P\left\{ \frac{S_{1,000,000} - 50,000}{\sqrt{60,000}} \geq \frac{49,000 - 50,000}{\sqrt{60,000}} \right\} \quad (4) \]
\[ \approx 1 - \Phi\left( \frac{49,000 - 50,000}{\sqrt{60,000}} \right), \quad (5) \]
since \( n = 1,000,000 \) is so large the random variable \( (S_{1,000,000} - 50,000)/\sqrt{60,000} \) essentially has the standard normal distribution. The answer is 0.999978.

3. We have
\[ \nu = \mu - 1/2 \sigma^2 = 0.2 - 1/2 0.16 = 0.12. \quad (6) \]

(a)
\[ E[\ln S(1)] = 0.12; \quad \text{Stdev [ln } S(1)] = 0.40. \quad (7) \]

(b)
\[ E[S(1)] = e^{0.2(1)} = 1.22; \quad (8) \]
\[ \text{Stdev [S(1)]} = \sqrt{E[S(1)^2] - (E[S(1)])^2} \quad (9) \]
\[ = \sqrt{e^{(2\nu+1/2 2^2\sigma^2)(1)} - (1.22)^2} \quad (10) \]
\[ = \sqrt{e^{(2\nu+2\sigma^2)(1)} - (1.22)^2} = 0.5088. \quad (11) \]
(c) We have
\[
P\{S(0.25) \leq 1.40\} = P\left\{ \ln \frac{S(0.25)}{S(0)} \leq \ln \frac{1.40}{S(0)} \right\} = P\left\{ \frac{\ln \frac{S(0.25)}{S(0)} - (0.12)(0.25)}{(0.40)\sqrt{0.25}} \leq \frac{\ln \frac{1.40}{S(0)} - (0.12)(0.25)}{(0.40)\sqrt{0.25}} \right\}
\]
\[
= \Phi \left( \frac{\ln 1.40 - (0.12)(0.25)}{(0.40)\sqrt{0.25}} \right) = \Phi(1.5326) = 0.93727.
\]

(d) We have
\[
P\{ \max_{0 \leq \tau \leq 0.25} S(\tau) \leq 1.40 \} = P\left\{ \max_{0 \leq \tau \leq 0.25} \ln \frac{S(\tau)}{S(0)} \leq \ln \frac{1.40}{S(0)} \right\} = P\{M_{0.25} \leq \ln 1.4\},
\]
where the latter probability is associated with a Brownian Motion process $X$ with drift $\mu = 0.03$ and $\sigma = 0.40$. Using equation (20) of the notes with $y = \ln 1.4$ and $t = 0.25$ the answer is 0.88185.

(e) The event that it will take longer than 1 year for the price to double (to a value of 2) is equivalent to the event that the maximum price over the year does not exceed 2, i.e., $P\{M_1 \leq 2\}$. Using equation (20) of the notes with $y = \ln 2$ and $t = 1$ the answer is 0.86454.

(f) We use equation (22) of the notes with $x = 0$, $b = \ln 2$ and $a = \ln 0.5$. The answer is 0.73880.

4. (a) $d_1 = 0.641287$ and $d_2 = 0.512188$.
   (b) $\Phi(d_1) = 0.739286$ and $\Phi(d_2) = 0.695688$.
   (c) $C = 5.79794$.

5. We have
\[
u = e^{\sigma \sqrt{\Delta t}} = 1.105, \quad d = 1/u = 0.905, \quad p = \frac{1}{2} + \frac{1}{2} \frac{\nu}{\sigma \sqrt{\Delta t}} = 0.65.
\]
The last period’s node values (from top to bottom) are 149.182, 122.140, 100, 81.873, and 67.032. The probabilities follow the binomial distribution. For example,
\[
P(S_4 = 122.140) = P(3 \text{ “up” and 1 “down” price movements}) = \binom{4}{3} p^3 (1-p)^1 = 0.384
\]
\[
P(S_4 = 81.873) = P(1 \text{ “up” and 3 “down” price movements}) = \binom{4}{1} p^1 (1-p)^3 = 0.111.
\]
6. (a) \( C = 5.5623, \Delta = 0.8254, \Gamma = 0.04235 \) and \( \Theta = -4.55831 \).

(b) i. \( dC \approx (0.8254)(1) + (0.5)(0.04235)(1)^2 + (-4.55831)(1/26) = 0.67126 \Rightarrow \text{new } C \approx 6.23356 \).

ii. New \( C = 6.23246 \).