Options Concepts Homework III
ISyE 4803C (Fall 06)

1. Consider a family of call options on a non-dividend paying stock, each option being identical except for its strike price. The value of the call with strike price $K$ is denoted by $C(K)$. Prove the following two general relations using arbitrage arguments:

(a) If $K_2 > K_1$, then
\[ K_2 - K_1 \geq C(K_1) - C(K_2). \]

(b) If $K_3 > K_2 > K_1$, then
\[ C(K_2) \leq \left( \frac{K_3 - K_2}{K_3 - K_1} \right) C(K_1) + \left( \frac{K_2 - K_1}{K_3 - K_1} \right) C(K_3). \]

2. Let $S_t$ denote the price of a non-dividend paying stock. The price process of $S$ follows a binomial lattice with $u = 1.5$ and $d = 0.9$. $R = 1.1$ and $S_0 = 100$. Let $C(T, K)$ denote a European call option on $S$ with a maturity of $T$ years and strike price of $K$.

(a) Determine the time 0 price of $C(2, 80)$.
(b) Determine the time 0 price of $C(2, 110)$.
(c) Determine the cost of the portfolio of $6C(2, 80) + 3C(2, 110)$.

3. Consider the two-period real estate option of Exam I. Let $P_t$ denote the price of a one-unit condominium at time $t$.

(a) The value of this real estate option is provided in the solutions to Exam I. Compare it to your answer to Question 2(c) above.
(b) Let $V_2(P_2)$ denote the final payoffs of this option. Write a formula for $V_2(P_2)$. Here, $P_2 \in [0, \infty)$.
(c) Graphically depict the function $V_2(\cdot)$ (i.e. label the $x$-axis $P_2$ and $y$-axis $V_2(\cdot)$). Here, $P_2 \in [0, \infty)$.
(d) Use (c) to establish your answer to (a).

4. Consider a non-dividend paying stock $S$ whose price process follows a binomial lattice with $u = 2$ and $d = 0.5$. $R = 1.25$ and $S_0 = 4$. Define
\[ Y_t := \sum_{k=0}^{t} S_k, \quad t = 0, 1, 2, 3 \]

to be the sum of the stock prices between times zero and $t$. Consider a (European) Asian call option that expires at time three and has a strike price $K = 4$; that is, its payoff at time three is
\[ \max \left\{ \frac{Y_3}{4} - 4, 0 \right\}. \]
Table 1: A stochastic volatility, random interest rate model (Problem 5)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>4, $r_0 = 25%$</td>
<td>$S_1(U) = 8, r_1(U) = 25%$</td>
<td>$S_2(UU) = 12$</td>
</tr>
<tr>
<td></td>
<td>$S_1(D) = 2, r_1(D) = 50%$</td>
<td>$S_2(UD) = S_2(DU) = 8$</td>
<td>$S_2(DD) = 2$</td>
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This is like a European call option, except the payoff of the option is based on the average stock price rather than the final stock price. Let $V_t(s, y)$ denote the price of this option at time $n$ if $S_t = s$ and $Y_t = y$. In particular,

$$V_3(s, y) = \max\{y/4 - 4, 0\}.$$

(a) Develop an algorithm for computing $V_t$ recursively. In particular, write a formula for $V_t$ in terms of $V_{t+1}$.

(b) Apply the algorithm developed in (a) to compute $V_0(4, 4)$, the price of the Asian option at time zero.

(c) Provide a formula for $\delta_t(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time $t$ if $S_t = s$ and $Y_t = y$.

5. In this problem we consider a two-period, stochastic volatility, random interest rate model. The stock prices and interest rates are provided in Table 1. Consider the European option whose final payoffs are $V_2 = \max\{S_2 - 7, 0\}$. Determine the value of this option at times 0 and 1.

6. We consider the Complexico infinite horizon mining problem when the profit flow $c(x, z)$ is given by $(x - z)z$ instead of $gz - 500z^2/x$. Let $d$ denote the one-period discount factor. Let $V(x)$ denote the optimal value function, and let $z(x)$ denote the optimal production quantity.

(a) Write down the Bellman equation for $V(x)$ using the principle of optimality.

(b) Explain why $V(x)$ cannot be linear.

(c) Determine $V(x)$ when $d = 0$.

(d) Use the method of successive approximations to determine the functional form of $V(x)$ and $z(x)$.

(e) Given the functional form for $V(x)$, determine the exact value function and optimal policy by finding the fixed point. (This involves solving an algebraic equation.)