Mean-Variance Portfolio Analysis  
and the  
Capital Asset Pricing Model  

1 Introduction  

In this handout we develop a model that can be used to determine how a risk-averse investor can choose an optimal asset portfolio in this sense:  

- the investor will earn the highest possible expected return given the level of volatility the investor is willing to accept or, equivalently,  
- the investor’s portfolio will have the lowest level of volatility given the level of expected return the investor requires.  

The techniques used are called mean-variance optimization and the underlying theory is called the Capital Asset Pricing Model (CAPM). Under the assumptions of CAPM, it is possible to determine the expected “risk-adjusted” return of any asset/security, which incorporates the security’s expected return, volatility and its correlation with the “market portfolio.”
2 Market Setup

We consider a market with \( n \) risky assets, \( i = 1, 2, \ldots, n \) and a risk-free asset labeled 0. An investor wishes to invest \( B \) dollars in this market. Let

- \( B_i, i = 0, 1, 2, \ldots, n \), denote the allocation of the budget to asset \( i \) so that \( \sum_{i=0}^{n} B_i = B \),
- \( x_i := B_i/B \) denote the portfolio weight of asset \( i \), namely, the fraction of the investor’s budget allocated to asset \( i \),
- \( R_i \) denote the random one-period return on asset \( i, i = 1, 2, \ldots, n \), and let
- \( r_f \) denote the risk-free return.

The investor’s one-period return on his/her portfolio is given by

\[
\text{One-period return} = \frac{\sum_{i=0}^{n} B_i R_i}{B} = \sum_{i=0}^{n} \frac{B_i}{B} R_i = \sum_{i=0}^{n} x_i R_i. \tag{1}
\]

We shall think of \( B \) as fixed and hereafter identify a portfolio of the \( n+1 \) assets with a vector

\[
x = (x_0, x_1, x_2, \ldots, x_n) \text{ such that } \sum_{i=0}^{n} x_i = 1. \tag{2}
\]

The portfolio’s random return will be denoted by

\[
R_P = R(x) := \sum_{i=1}^{n} x_i R_i. \tag{3}
\]

We shall use the symbols ‘\( x \)’ or \( P \) to refer to a portfolio.

When \( x_i < 0 \) the holder of the portfolio is short-selling asset \( i \). In this handout we permit unlimited short-selling. In practice, however, there are limits to the magnitude of short-selling. If short-selling is not permitted, then the solution approach outlined in these notes does not directly apply, though the problem can be easily solved with commercial software.
Example 1  Consider a portfolio of 200 shares of firm A worth $30/share and 100 shares of firm B worth $40/share. The total value of the portfolio is
\[ 200(30) + 100(40) = 10,000. \]  
(4)
The respective portfolio weights are
\[ x_A = \frac{200(30)}{10,000} = 60\%, \quad x_B = \frac{100(40)}{10,000} = 40\%. \]  
(5)

Example 2  Suppose you bought the portfolio of Example 1, and suppose further that firm A’s share price goes up to $36 and firm B’s share price falls to $38.

- What is the new value of the portfolio?

The new value of the portfolio is
\[ 200 \times (36) + 100(38) = 11,000. \]  
(6)

- What return did this portfolio earn?

The portfolio’s gain was $1,000 or 10% return on investment.
A’s return was \( \frac{36}{30} - 1 = 20\% \) and B’s return was \( \frac{38}{40} - 1 = -5\% \).

Since the initial portfolio weights are \( x_A = 60\% \) and \( x_B = 40\% \), we can also compute the portfolio’s return as
\[ R_P = x_A R_A + x_B R_B = 0.60(20\%) + 0.40(-5\%) = 10\%. \]  
(7)

- After the price change, what are the new portfolio weights?

The new portfolio weights are
\[ x_A = \frac{200(36)}{11,000} = 65.45\%, \quad x_B = \frac{100(38)}{11,000} = 34.55\%. \]  
(8)
3 Portfolio Return

The expected return of a portfolio $P$ is given by

$$
\mu(x) := E[R_P] = E[\sum_i x_i R_i] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i].
$$

(9)

Example 3 You invest $1000 in stock A, $3000 in stock B. You expect a return of 10% for stock A and 18% in stock B.

- What is your portfolio’s expected return?

Your portfolio weights are $x_A = 1,000/4,000 = 25\%$ and $x_B = 3,000/4,000 = 75\%$. Therefore,

$$
E[R_P] = 0.25(10\%) + 0.75(18\%) = 16\%.
$$

(10)
4 Portfolio Variance: Basic definitions and properties

Let

\[ \bar{R}_i := E[R_i] \]  \hspace{1cm} (11) \\
\[ \text{Var}[R_i] := E((R_i - \bar{R}_i)^2) = E[R_i^2] - (\bar{R}_i)^2 \]  \hspace{1cm} (12) \\
\[ \sigma_i := \sqrt{\text{Var}[R_i]} \]  \hspace{1cm} (13)

denote, respectively, the mean, variance and standard deviation of \( R_i \).

The covariance between two random variables \( X \) and \( Y \) is defined as

\[ \text{Cov}(X,Y) = \sigma_{XY} := E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y], \]  \hspace{1cm} (14)

and the correlation between two random variables \( X \) and \( Y \) is defined as

\[ \text{Corr}(X,Y) = \rho_{XY} := \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \]  \hspace{1cm} (15)

A correlation value must always be a number between -1 and 1. The covariance can be determined from correlation via

\[ \sigma_{XY} = \text{Cov}(X,Y) = \text{Corr}(X,Y) \sigma_X \sigma_Y = \rho_{XY} \sigma_X \sigma_Y. \]  \hspace{1cm} (16)

It follows directly from (14) that

- variance may be expressed in terms of covariance:
  \[ \text{Var}(X) = \text{Cov}(X,X), \]  \hspace{1cm} (17)

- covariance is symmetric:
  \[ \text{Cov}(X,Y) = \text{Cov}(Y,X), \]  \hspace{1cm} (18)

- covariance is bilinear:
  \[ \text{Cov}(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j), \]  \hspace{1cm} (19)
  \[ \text{Cov}(\alpha X, Y) = \text{Cov}(X, \alpha Y) = \alpha \text{Cov}(X,Y), \text{ a real number.} \]  \hspace{1cm} (20)
5 Portfolio Variance Formula

Let
\[ \sigma_{ij} := \text{Cov}(R_i, R_j) = \sigma_i \sigma_j \text{Corr}(R_i, R_j) = \sigma_i \sigma_j \rho_{ij} \] (21)
denote the covariance between the returns of assets \( i \) and \( j \), and let \( \Sigma \) denote the \( n \) by \( n \) matrix whose \((i,j)\)th entry is given by \( \sigma_{ij} \).

The matrix \( \Sigma \) is called the covariance matrix. Due to (18), \( \sigma_{ij} = \sigma_{ji} \), which implies that the covariance matrix is symmetric, namely, \( \Sigma = \Sigma^T \) (here, the symbol ‘T’ denotes transpose).

Using (17) and repeated application of (19) and (20), it can be shown that the portfolio’s (return) variance is
\[ \text{Var}[R(x)] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j. \] (22)

**Special Case: \( n = 2 \).** Here, the portfolio’s variance is
\[ \text{Var}[R(x)] = \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} x_i x_j = \sigma_1^2 x_1^2 + \sigma_{12} x_1 x_2 + \sigma_{21} x_1 x_2 + \sigma_2^2 x_2^2 \] (23)
\[ = \sigma_1^2 x_1^2 + \sigma_{21}^2 x_2^2 + 2 \sigma_{12} x_1 x_2 \] (24)
\[ = \sigma_1^2 x_1^2 + \sigma_{21}^2 x_2^2 + 2 \sigma_1 \sigma_{21} \rho_{12} x_1 x_2 \] (25)
The last line uses the fact that \( \sigma_{12} \) and \( \sigma_{21} \) are equal.

**General Case.** In matrix notation, the portfolio’s variance may be represented as
\[ \text{Var}[R(x)] = x^T \Sigma x \] (26)
\[ = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \] (27)
\[ = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 x_1^2 + \sigma_{12} x_2 \\ \sigma_{21} x_1 + \sigma_2^2 x_2 \end{pmatrix} \] (28)
\[ = (x_1 \sigma_1^2 x_1 + x_1 \sigma_{12} x_2) + (x_2 \sigma_{21} x_1 + \sigma_2^2 x_2) \] (29)
\[ = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2 \sigma_{12} x_1 x_2. \] (30)
Example 4 Stock returns will be more highly correlated when they are similarly affected by the same economic events. This is why stocks in the same industry tend to have highly correlated returns than stocks in somewhat different industries. Table 1 (Table 11.3, p. 336 in Corporate Finance by Berk and DeMarzo) provides some examples.

Table 1: Historical Annual Volatilities and Correlations for Selected Stocks (based on monthly returns, 1996-2008).

<table>
<thead>
<tr>
<th></th>
<th>Microsoft</th>
<th>Dell</th>
<th>Alaskan</th>
<th>Southwest Airlines</th>
<th>Ford Motor</th>
<th>General Motors</th>
<th>General Mills</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility (StDev)</td>
<td>37%</td>
<td>50%</td>
<td>38%</td>
<td>31%</td>
<td>42%</td>
<td>41%</td>
<td>18%</td>
</tr>
<tr>
<td>Correlation with:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microsoft</td>
<td>1.00</td>
<td>0.62</td>
<td>0.25</td>
<td>0.23</td>
<td>0.26</td>
<td>0.23</td>
<td>0.10</td>
</tr>
<tr>
<td>Dell</td>
<td>0.62</td>
<td>1.00</td>
<td>0.19</td>
<td>0.21</td>
<td>0.31</td>
<td>0.28</td>
<td>0.07</td>
</tr>
<tr>
<td>Alaska Air</td>
<td>0.25</td>
<td>0.19</td>
<td>1.00</td>
<td>0.30</td>
<td>0.16</td>
<td>0.13</td>
<td>0.11</td>
</tr>
<tr>
<td>Southwest Airlines</td>
<td>0.23</td>
<td>0.30</td>
<td>1.00</td>
<td>0.25</td>
<td>0.22</td>
<td>0.22</td>
<td>0.20</td>
</tr>
<tr>
<td>Ford Motor</td>
<td>0.26</td>
<td>0.31</td>
<td>0.16</td>
<td>0.25</td>
<td>1.00</td>
<td>0.62</td>
<td>0.07</td>
</tr>
<tr>
<td>General Motors</td>
<td>0.23</td>
<td>0.28</td>
<td>0.13</td>
<td>0.22</td>
<td>0.62</td>
<td>1.00</td>
<td>0.02</td>
</tr>
<tr>
<td>General Mills</td>
<td>0.10</td>
<td>0.07</td>
<td>0.11</td>
<td>0.20</td>
<td>0.07</td>
<td>0.02</td>
<td>1.00</td>
</tr>
</tbody>
</table>

- What is the covariance between the returns for Microsoft and Dell?

\[
\text{Cov}(R_M, R_D) = \sigma_m \sigma_D \rho_{MD} = (0.37)(0.50)(0.62) = 0.1147. \tag{31}
\]

- What is the standard deviation of a portfolio with equal amounts invested in these two stocks?

\[
\begin{align*}
\text{Var}(R_P) & = \text{Var}(0.5R_m + 0.5R_D) \tag{32} \\
& = \sigma_m^2 x_M^2 + \sigma_D^2 x_D^2 + 2\sigma_{MD} x_M x_D \tag{33} \\
& = (0.37)^2(0.5) + (0.5)^2(0.5) + 2(0.1147) = 0.1541 \tag{34} \\
\sigma_P & = \sqrt{0.1541} = 39.26\%. \tag{35}
\end{align*}
\]
Example 5 Consider a portfolio of Intel and Coca-Cola stocks.

- Intel’s expected return is 26% and its volatility is 50%.
- Coca-Cola’s expected return is 6% and its volatility is 25%.
- The stock returns are independent, and so the correlation coefficient $\rho = 0$.

Suppose you have $20,000 in cash to invest. You decide to short sell $10,000 worth of Coca-Cola stock and invest the proceeds from your short sale plus your $20,000 in Intel.

- What are the portfolio weights?

Your short sale is a negative investment of $10,000 in Coca-Cola stock. You have invested $30,000 in Intel. Your total investment is still $20,000. Your portfolio weights are 1.5 or 150% in Intel and -0.5 or -50% in Coca-Cola.

- What is the portfolio’s expected return?

The expected portfolio return is

$$1.5(0.26) + (-0.5)(0.06) = 36\%.$$  \hspace{1cm} (36)

- What is the portfolio’s volatility?

The variance of the portfolio return (25) is

$$0.50^2(1.5)^2 + 0.25^2(-0.5)^2 = 0.578125.$$  \hspace{1cm} (37)

The standard deviation or volatility of the portfolio’s return is

$$\sqrt{0.578125} = 0.76 = 76\%.$$  \hspace{1cm} (38)

By short-selling you have dramatically increased the expected return but you have also significantly increased the volatility.
6 Minimum Variance Portfolio of Two Assets

The minimum variance portfolio achieves the lowest variance, regardless of expected return. For two assets, it is easy to solve for the minimum variance portfolio. (We shall see that it is also easy to solve for the minimum variance portfolio for the general case.)

From the two-asset portfolio variance formula (24) with \( x_2 = 1 - x_1 \), the variance is

\[
Var(x_1R_1 + (1 - x_1)R_2) = \sigma_1^2 x_1^2 + \sigma_2^2 (1 - x_1)^2 + 2x_1(1 - x_1)\sigma_{12}.
\]  

(39)

Since there are no constraints on the values for \( x_1 \), we can take the derivative with respect to \( x_1 \) and set equal to zero:

\[
0 := \frac{d}{dx_1} Var(R(x_1)) = 2\sigma_1^2 x_1 - 2(1 - x_1)\sigma_2^2 + 2\sigma_{12}(1 - 2x_1) \tag{40}
\]

\[
= 2(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})x_1 - 2(\sigma_2^2 - \sigma_{12}). \tag{41}
\]

Thus,

\[
x_1^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \quad x_2^* = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}. \tag{42}
\]
**Example 6** Let’s return to the data for the Intel and Coca-Cola stocks. Intel’s volatility is 50% and Coca-Cola’s volatility is 25%. Here, we assume the correlation coefficient is 0.20.

- What is the covariance between the returns of these two stocks?

  The covariance \( \sigma_{12} = (0.50)(0.25)(0.2) = 0.025 \).

- What are the portfolio weights of the minimum variance portfolio of these two stocks?

  The optimal weight for the Intel stock is
  \[
  \frac{(0.25)^2 - 0.025}{(0.50)^2 + (0.25)^2 - 2(0.025)} = \frac{0.0375}{0.2625} = 1/7, \tag{43}
  \]
  and thus the optimal weight for the Coca-Cola stock is 6/7.

- What is the minimum variance portfolio’s volatility?

  \[
  Var(R(x^*)) = (0.50)^2(1/7)^2 + (0.25)^2(6/7)^2 + 2(1/7)(6/7)(0.025) \tag{44}
  \]
  \[
  = 0.05714. \tag{45}
  \]
  \[
  StDev(R(x^*)) = \sqrt{0.05714} = 0.239 = 23.9\%. \tag{46}
  \]

  Note how the minimum volatility is less than the volatility of either stock!
7 Minimum Variance Portfolio of Risky Assets

Let $e^T := (1, 1, \ldots, 1)$ denote the $n$-vector whose components are all ones. It can be shown that the minimum variance portfolio $x^*$ satisfies this equation:

$$
x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \lambda \left( \Sigma^{-1} e \right).
$$

(48)

- The vector $x^*$ is proportional to the vector $\Sigma^{-1} e$, with $\lambda$ being the proportionality constant.
- The multiplication of $\Sigma^{-1}$ by the vector $e$ simply computes the vector of row sums of the inverse of the covariance matrix $\Sigma^{-1}$.
- Since $\sum_i x_i^* = 1$, the proportionality constant $\lambda$ is simply the reciprocal of the sum of the components of $\Sigma^{-1} e$. So, to arrive at the optimal solution $x^*$ from $\Sigma^{-1} e$ all one has to do is simply “normalize” the latter vector by its sum. For example, suppose the vector $\Sigma^{-1} e = (10, 20, 30)$. The sum is 60 and so the unique vector that is proportional to $(10, 20, 30)$ and which sums to one is the vector $(1/6, 1/3, 1/2)$.
- Equivalently, the proportionality constant $\lambda = (e^T \Sigma^{-1} e)^{-1}$. In words, the value of $\lambda$ can be obtained by merely summing up the values for all elements of the inverse of the covariance matrix, and taking the reciprocal.
- $\lambda$ actually equals the minimum variance!
**Example 7** Let’s return to the Intel, Coca-Cola data of Example 6. Intel’s volatility is 50%, Coca-Cola’s volatility is 25% and the correlation coefficient is 0.20. Recall that we found that the minimum variance portfolio was (1/7, 6/7). In this example,

\[
\Sigma = \begin{pmatrix}
0.2500 & 0.0250 \\
0.0250 & 0.0625 \\
\end{pmatrix},
\]

and

\[
\Sigma^{-1} = \frac{1}{(0.25)(0.0625) - (0.025)^2} \begin{pmatrix}
0.0625 & -0.0250 \\
-0.0250 & 0.2500 \\
\end{pmatrix}
\]

\[
= \frac{1}{0.015} \begin{pmatrix}
0.0625 & -0.0250 \\
-0.0250 & 0.2500 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
4.16 & -1.6 \\
-1.6 & 16.6 \\
\end{pmatrix}.
\]

The vector of row sums associated with \(\Sigma^{-1}\) is (2.5, 15). The sum of these two numbers 17.5. Thus,

- The minimum variance portfolio vector is (2.5/17.5, 15/17.5) = (1/7, 6/7), as before!
- The minimum variance equals 1/17.5 and the minimum standard deviation equals \(\sqrt{1/17.5} = 23.9\%\), as before!

**Useful Tip.**

- If you are just interested in the minimum variance portfolio, you do not need to calculate the determinant here as it is just a proportionality constant. (You have to normalize the vector \(\Sigma^{-1}e\) anyway, so the constant 1/0.015 is unnecessary.)
- Keep in mind that you will need the determinant to correctly calculate the minimum variance. Alternatively, you can use (25) directly to calculate the variance once you have the minimum variance portfolio.
Example 8 Consider three risky assets whose covariance matrix $\Sigma$ is

$$
\Sigma = \begin{pmatrix}
216 & 70 & -324 \\
70 & 25 & -150 \\
-324 & -150 & 1596
\end{pmatrix}.
$$

(Here, the returns were measured in percents.) The inverse of the covariance matrix is

$$
\Sigma^{-1} = \begin{pmatrix}
0.1480 & -0.5367 & -0.0204 \\
-0.5367 & 2.0388 & 0.0827 \\
-0.0204 & 0.0827 & 0.0043
\end{pmatrix}.
$$

Thus,

$$
\Sigma^{-1}e = \begin{pmatrix}
-0.4081 \\
1.5971 \\
0.0661
\end{pmatrix}
$$

$$
e^T\Sigma^{-1}e = -0.4081 + 1.5971 + 0.0661 = 1.2379.
$$

- Dividing the vector $(-0.4081, 1.5971, 0.0661)$ by $1.2379$ yields $(-0.3297, 1.2756, 0.0534)$, and this is the minimum variance portfolio.

- The sum of all entries of $\Sigma^{-1}$ is $1.2379$, and this value equals the reciprocal of the minimum variance. Consequently, the minimum variance portfolio’s volatility is $\sqrt{1/1.2379} = 0.8078 = 80.78\%$. 

8 Volatility of a Large Portfolio: The benefits of diversification

Let $\text{AvgVar}$ denote the average variance of the individual stocks, and let $\text{AvgCov}$ denote the average covariance between the stocks. Imagine a world where the variance of each stock was constant and equal to $\text{AvgVar}$ and the covariance between stocks was constant and equal to $\text{AvgCov}$.

Now consider an equally-weighted portfolio of these stocks in which $x_i = 1/n$ for each $i = 1, 2, \ldots, n$. What is the variance of this equally-weighted portfolio? The covariance matrix $\Sigma$ has $n^2$ entries, $n$ along the diagonal and $n^2 - n = n(n - 1)$ off-diagonal entries. The entries along the diagonal correspond to each stock’s variance which equals $\text{AvgVar}$, whereas the off-diagonal entries correspond to the covariances which each equal $\text{AvgCov}$. Therefore, in this special setting

$$\text{Var}(R_P) = \frac{\text{AvgVar}}{n} + \left(1 - \frac{1}{n}\right) \times \text{AvgCov}. \quad (57)$$

We can see that $\text{Var}(R_P) \to \text{AvgCov}$ as $n \to \infty$. The limiting portfolio standard deviation $\text{StDev}(R_P) = \sqrt{\text{AvgCov}}$.

The historical volatility (standard deviation) of the return of a large stock is about 40% and the typical correlation between the returns of large stocks is about 28%. On average, then,

$$\text{AvgVar} = (0.40)^2 = 0.16, \quad (58)$$
$$\text{AvgCov} = (0.40)(0.40)(0.28) = 0.0448, \quad (59)$$
$$\text{StDev}(R_p) = \sqrt{0.16/n + 0.0448(1 - 1/n)} \to 21.17\%. \quad (60)$$

Note that when $n = 20$ the $\text{StDev}(R_p) = 22.49\%$, which is very close to the limiting volatility.

What about a portfolio with arbitrary weights? It can be shown that

$$\text{Var}(R_P) = \sigma_P^2 = \sum_i x_i \sigma_i \sigma_P \text{Corr}(R_i, R_P). \quad (61)$$

Dividing both sides of this equation by $\sigma_P$ yields this very important decomposition of the volatility of a portfolio:

$$\text{StDev}(R_P) = \sum_i \left( \frac{x_i \sigma_i}{\sigma_P} \times \frac{\sigma_i}{\text{Total risk of } i} \times \frac{\text{Corr}(R_i, R_P)}{\text{Fraction of } i's \text{ risk common to } P} \right). \quad (62)$$

Assume that not all assets have a perfect positive correlation of +1 and that the weights are all positive. We can conclude that the volatility of the portfolio will be lower than the weighted average volatility given by $\sum_i x_i \sigma_i$. Note how the expected return of a portfolio $\sum_i x_i E[R_i]$ is the weighted average of the individual returns. So, you can eliminate some of the volatility by diversifying.
9 Mean-Variance Efficient Frontier of Risky Assets

Consider once again the Intel and Coca-Cola stocks. Let $x$ denote the weight on Intel and $1-x$ denote the weight on Coca-Cola. As $x$ varies from 0 to 1, the portfolio’s mean return $E[R(x)]$ and standard deviation $StDev(R(x)) = \sqrt{Var(R(x))}$ will vary. The points 

$$\{(StDev(R(x)), E[R(x)]) := \{\sigma(x), \mu(x)\}$$

trace out a curve whose shape critically depends on the correlation coefficient $\rho$.

Questions: What is the shape of the curve when $\rho = +1$? $\rho = -1$? How about values for $\rho$ in between?

* Expected return. The expected return

$$E[R(x)] = 0.26x + 0.06(1-x) = 0.06 + 0.20x$$

is linear in $x$ and does not depend on $\rho$.

* Variance.

$$Var(R(x)) = (0.5)^2 x^2 + (0.25)^2 (1-x)^2 + 2x(1-x)(0.50)(0.25)\rho$$

The variance is a quadratic function of $x$.

Let’s consider the special case when $\rho = +1$:

$$Var(R(x)) = (0.5)^2 x^2 + (0.25)^2 (1-x)^2 + 2x(1-x)(0.50)(0.25) = (0.5x + 0.25(1-x))^2$$

$$StDev(R(x)) = 0.25 + 0.25x, \ x \in [0,1]$$

When $\rho = +1$, the $StDev$ is linear in $x$.

Now let’s consider the special case when $\rho = -1$:

$$Var(R(x)) = (0.5)^2 x^2 + (0.25)^2 (1-x)^2 - 2x(1-x)(0.50)(0.25) = (0.5x - 0.25(1-x))^2$$

$$StDev(R(x)) = |0.75x - 0.25|, \ x \in [0,1].$$

When $\rho = -1$, the $StDev$ is “piecewise linear” and will actually be zero when $x = 1/3$!
Let’s consider the general case where \( x = (x_1, x_2, \ldots, x_n) \) denotes a portfolio of risky assets. For each choice of \( x \) we can compute the values for its (volatility, mean) = \((\sigma(x), \mu(x))\). Imagine that we plot these points in the \((\sigma, \mu)\) plane for all possible portfolio’s \( x \). Which portfolio’s will a risk-averse investor choose?

Fix a value for the expected return, say \( \mu = 10\% \), and consider all portfolio’s \( x \) such that \( \mu(x) = 10\% \). Among all such portfolio’s it makes sense to choose one that minimizes the volatility. Let’s call this minimum volatility \( \sigma(\mu) \). As we choose different values for \( \mu \) we obtain different values for \( \sigma(\mu) \). In fact, the collection of all such points \( \{(\sigma(\mu), \mu)\} \) in the \((\sigma, \mu)\) plane trace out a “boundary curve.”

We can also fix a value for the volatility, \( \sigma = 20\% \), and consider all portfolio’s \( x \) such that \( \sigma(x) = 20\% \). Among all such portfolio’s it makes sense to choose one that maximizes the expected return. Let’s call this maximum expected return \( \mu(\sigma) \). As we choose different values for \( \sigma \) we obtain different values for \( \mu(\sigma) \). In fact, the collection of all such points \( \{(\sigma, \mu(\sigma))\} \) in the \((\sigma, \mu)\) plane trace out a “boundary curve,” too. This curve will be identical to the previous curve.

This boundary curve is called the mean-variance efficient frontier of risky assets.
10 Mean-Variance Efficient Frontier of All Assets

There is also a risk-free asset. Consider a portfolio of the risk-free asset and a risky asset $k$ whose expected return is $\bar{R}_k$ and whose volatility is $\sigma_k$. This risky asset could be a portfolio of the underlying risky assets. Let $x$ denote the weight on the risky asset with $1 - x$ being the weight on the risk-free asset. We know that

$$\mu(x) = x\bar{R}_k + (1 - x)r_f$$

$$\sigma(x) = \sqrt{x^2\sigma_k^2 + (1 - x)^2\sigma_0^2 + 2x(1 - x)\sigma_k0\rho_{ij}} = x\sigma_k + (1 - x)\sigma_0.$$  

Observe that

$$(\sigma(x), \mu(x)) = x(\bar{R}_k, \sigma_k) + (1 - x)(r_f, 0).$$

Consequently, as $x \geq 0$ varies, the collection of all points $(\sigma(x), \mu(x))$ in the $(\sigma, \mu)$ plane traces out a ray emanating from the point $(0, r_f)$ whose slope equals $(\bar{R}_k - r_f)/\sigma_k$. Of course, there are many such risky portfolio’s $k$. By combining each one with the risk-free asset, more $(\sigma, \mu)$ points are added.

**Key Observation.**

If the slope of the ray associated with risky asset $j$ is larger than the slope of the ray associated with risky asset $k$, then the points on the ray associated with $j$ dominate those on the ray associated with $k$. Consequently, there is only one risky portfolio “T” that matters, namely, the one whose ray is tangent to the mean-variance efficient frontier of risky assets!

This portfolio is called the tangent (or market) portfolio. It solves the following optimization problem:

$$(TP) : \max_x \frac{\mu(x) - r_f}{\sigma(x)}.$$  

The objective function is called the Sharpe ratio.
11 Application of the Sharpe Ratio

The Sharpe ratio can be used to decide whether adding an asset to a portfolio will be beneficial. Here is the setup. You are given a portfolio $P$ and an asset $i$ not in portfolio $P$. Will adding asset $i$ improve your portfolio? Adding asset $i$ (either with a positive or negative weight) to $P$ will be beneficial if by doing so it will increase the (new) Sharpe ratio.

**Example 9** Suppose portfolio $P$’s expected return is 12%, its volatility is 20% and the risk-free rate is 3%. Suppose further that a particular mix of asset $i$ and $P$ yields a portfolio $P'$ with an expected return of 18% and a volatility of 30%.

- What is the Sharpe ratio of $P$?

  The Sharpe ratio of $P$ is $(0.12-0.03)/0.20 = 0.45$.

- What is the Sharpe ratio of $P'$?

  The Sharpe ratio for $P'$ is $(0.18-0.03)/0.30 = 0.50$.

- Show that asset $i$ is in fact beneficial.

  Form a portfolio of $P'$ and the risk-free asset with respective weights 2/3 and 1/3. This new portfolio will have a volatility of $2/3(0.30) = 0.20 = 20\%$, the same as portfolio $P$. This new portfolio’s expected return is $2/3(18\%) + 1/3(3\%) = 13\%$. Thus, this new portfolio has the same volatility but with a higher expected return than $P$.

  Alternatively, form a portfolio of $P'$ and the risk-free asset with respective weights 60% and 40%. This new portfolio will have an expected return of $0.60(18\%) + 0.40(3\%) = 12\%$, the same as portfolio $P$. This new portfolio’s volatility is $0.60(0.30) = 0.18 = 18\%$. Thus, this new portfolio has the same expected return but with a lower volatility than $P$.

18
12 Computing the Tangent Portfolio

All mean-variance efficient portfolios must be a combination of the risk-free security and the tangent portfolio. That is, each investor should allocate a portion of their budget to the risk-free security and the remaining portion to the tangent portfolio. The sub-allocations to the individual risky securities are determined by the weights \( x^* \) that define the tangent portfolio (to be determined). A very risk-averse investor allocates very little to the tangent portfolio, whereas an investor with a large appetite for risk may hold a negative portion in the risk-free security (i.e. borrow at the risk-free rate) to invest more than the budget into the tangent portfolio.

Define

\[
\hat{R} := \bar{R} - r_{fe}.
\] (77)

It is the vector of excess returns obtained by simply subtracting the risk-free rate from each of the asset’s expected returns. It can be shown that the tangent portfolio must be proportional to \( \Sigma^{-1}\hat{R} \), and thus it may be found in the analogous way we found the minimum variance portfolio.

Example 10 Consider again our three asset data of the previous example. Suppose the expected returns are \( (\bar{R}_1, \bar{R}_2, \bar{R}_3) = (18\%, 10\%, 8\%) \), and the risk-free rate is 3%.

- What is the vector of excess expected returns?

The vector of excess expected returns is \( (18\% - 3\%, 10\% - 3\%, 8\% - 3\%) = (15\%, 7\%, 5\%) \).

- What is the tangent portfolio?

The tangent portfolio is proportional to

\[
\Sigma^{-1} \begin{pmatrix} 15 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -1.6389 \\ 6.6346 \\ 0.2944 \end{pmatrix}.
\] (78)

Now divide the right-hand side of (78) by 5.2901 = −1.6389 + 6.6346 + 0.2944 to obtain the tangent portfolio as \( (-0.3098, 1.2542, 0.0557) \). You may verify that the tangent portfolio’s expected return is 7.4112% and its volatility equals \( \sqrt{0.8358} = 91.31\% \).

The fact that the tangent portfolio is proportional to \( \Sigma^{-1}\hat{R} \) implies a relationship between a security’s expected return and a measure of its risk. This relationship is called the Security Market Line (SML) and is the most widely used approach to determine an appropriate expected return for a security (or asset). The SML can also be used to decide whether an asset should be added to an existing portfolio and, if so, how to determine its weight in the new portfolio.
13 Security Market Line Derivation

Given a portfolio $P$ and an asset $i$ that may or may not already belong to portfolio $P$, we already know that the portfolio weight vector $w^* = (w^*_P, w^*_i)$ that maximizes the Sharpe ratio is proportional to the vector $\Sigma^{-1}\hat{R}$. Since

$$\Sigma = \begin{pmatrix} \sigma^2_P & \sigma_{iP} \\ \sigma_{iP} & \sigma^2_i \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} \hat{R}_P \\ \hat{R}_i \end{pmatrix},$$

it follows that

$$w^* = \begin{pmatrix} w^*_P \\ w^*_i \end{pmatrix} \propto \begin{pmatrix} \sigma^2_i & -\sigma_{iP} \\ -\sigma_{iP} & \sigma^2_P \end{pmatrix} \begin{pmatrix} \hat{R}_P \\ \hat{R}_i \end{pmatrix} = \begin{pmatrix} \sigma^2_P\hat{R}_i - \sigma_{iP}\hat{R}_P \\ \sigma^2_P\hat{R}_i - \sigma_{iP}\hat{R}_P \end{pmatrix}. \quad (80)$$

The only way asset $i$ will not be beneficial to portfolio $P$ is if

$$\sigma^2_P\hat{R}_i - \sigma_{iP}\hat{R}_P = 0, \quad (81)$$

which is equivalent to the condition that

$$\hat{R}_i = \frac{\sigma_{iP}}{\sigma^2_P}\hat{R}_P. \quad (82)$$

Define the parameter

$$\beta^P_i := \frac{\text{Cov}(\hat{R}_i, \hat{R}_P)}{\text{Var}(\hat{R}_P)} = \frac{\rho_{iP}\sigma_i\sigma_P}{\sigma^2_P} = \frac{\rho_{i}\sigma_i}{\sigma_P}. \quad (83)$$

It is called the asset’s beta (with respect to the portfolio $P$). Using the parameter $\beta^P_i$, we may conclude that it is not possible to improve the Sharpe ratio of $P$ with asset $i$ if and only if

$$\hat{R}_i = r_f + \beta^P_i(\hat{R}_P - r_f) := \text{required return for asset } i. \quad (84)$$

Conclusions.

- The right-hand side of (84) is the required return for asset $i$ to compensate for the risk it will contribute to portfolio $P$. If $\hat{R}_i$ is higher than its required return, then adding asset $i$ to $P$ with some positive weight will increase the Sharpe ratio; if $\hat{R}_i$ is lower than its required return, then adding asset $i$ to $P$ with some negative weight will increase the Sharpe ratio.

- The derivation above applies if asset $i$ already belongs to the portfolio $P$ with some non-zero weight. If asset $i$’s expected return does not equal its required return, this means that the existing weight $x_i$ in portfolio $P$ can be adjusted (up or down) to improve the Sharpe ratio.

- When portfolio $P$ is mean-variance efficient, its Sharpe ratio cannot be improved, which implies that existing portfolio weights cannot be adjusted without lowering the Sharpe ratio.
Example 11 You are currently invested in a broad-based fund with an expected return of 15% and a volatility of 20%. Your broker suggests that you add a real estate fund to your portfolio. It has an expected return of 9%, a volatility of 35% and a correlation of 0.10 with your existing fund. The risk-free rate is 3%.

- Will adding the real estate fund improve your portfolio?

Let $P$ denote the portfolio consisting of the existing fund and let $i$ denote the real estate fund. Here,

$$\beta_i^P = \rho_{iP} \sigma_i / \sigma_P = 0.10(0.35)/(0.20) = 0.175,$$

and so

$$r_f + \beta_i^P (\bar{R}_P - r_f) = 3\% + 0.175(15\% - 3\%) = 5.1\%.$$

As $5.1\% < 9\%$, it will certainly be advantageous to invest some positive amount in the real estate fund to improve your Sharpe ratio of $(0.15-0.03)/0.20 = 0.60$. 

Example 12 Consider the data for the previous example. What is the best combination of the broad-based and real estate funds to maximize the Sharpe ratio?

The covariance is $\sigma_{iP} = (0.35)(0.20)(0.1) = 0.007$, 

$$\Sigma = \begin{pmatrix} \sigma_P^2 & \sigma_{iP} \\ \sigma_{iP} & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} 0.04 & 0.007 \\ 0.007 & 0.1225 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} \hat{R}_P \\ \hat{R}_i \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}. \quad (87)$$

Consequently, 

$$w^* = \begin{pmatrix} w^*_P \\ w^*_i \end{pmatrix} \propto \begin{pmatrix} 0.1225 & -0.007 \\ -0.007 & 0.04 \end{pmatrix} \begin{pmatrix} 12 \\ 6 \end{pmatrix} = \begin{pmatrix} 1.428 \\ 0.156 \end{pmatrix}, \quad (88)$$

and so the optimal portfolio weights are 

$$w^* = (w^*_P, w^*_i) = \left( \frac{1.428}{1.428 + 0.15}, \frac{0.156}{1.428 + 0.15} \right) = (90.15\%, 9.85\%). \quad (89)$$

Suppose you had $10,000 invested in the broad-based portfolio. You could rebalance your portfolio in two ways:

- Keep the $10,000 in the broad-based portfolio. The optimal portfolio weights imply that for every dollar invested in the broad-based fund, you need to invest $0.0985/0.9015 = $0.1093 in the real estate fund. So you need to invest $1,093 in the real estate fund. You would borrow this money from at the risk-free rate.

- If you did not wish to borrow funds, you would then sell $985 of the broad-based portfolio and invest the proceeds into the real estate fund. You would then have $9,015 in the broad-based portfolio, $985 in the real-estate fund, thereby achieving the desired portfolio weights.
14 The Security Market Line with the Market Portfolio

When the portfolio $P$ consists of “all” risky assets, it is called the Market Portfolio. The symbol ‘P’ is typically replaced with the symbol ’M’. The S&P 500 or other broad-based index funds are used as a proxy for the Market Portfolio. The term $\bar{R}_M - r_f$ is called the Market Premium.

The Security Market Line states that

Security’s expected return = risk-free rate + (security’s beta)*(Market Premium),

(90)

or, equivalently,

\[
\hat{R}_i = \beta_i \hat{R}_M
\]

(91)

In the portfolio context it is the security’s $\beta$ that is considered an appropriate measure of a security’s “risk.”

This relationship is what economists call an equilibrium relationship in the following sense: If a security’s expected return was higher (lower) than what is predicted by the SML, then there would be more (less) demand for it, thereby raising (lowering) its price and ultimately lowering (raising) its expected return to place it back on the SML.

Based on the past 50 years, the market premium is approximately 5%. With a risk-free rate of 2% the SML becomes $\bar{R} = 0.02 + 0.05 \times \beta$.

- If $\beta = 0$, then $\bar{R} = 0.02$. So when a security is uncorrelated with the market, it only “deserves” an expected return equal to the risk-free rate. This is because you can diversify away any of its firm-specific risk with a large portfolio.

- If $\beta = 1$, then $\bar{R} = \bar{R}_M$. So when a security’s $\beta$ equals one, then its expected return should equal the market portfolio’s expected return.

- Suppose $\beta < 0$, i.e., a security is negatively correlated with the market portfolio? The SML tells us its expected rate of return should be lower than the risk-free rate! The reason for this is that this security provides an additional benefit by lowering the standard deviation of the market portfolio due to its negative correlation. This security acts like a “hedge.” Investors will accept a lower expected rate of return for such a security because it provides a benefit for lowering overall portfolio risk.
What is the beta of a portfolio $P$? It can be shown that

$$\beta_P = \sum_i x_i \beta_i. \quad (92)$$

That is, the beta of a portfolio is the weighted average of the beta of the assets in the portfolio.

**Example 13** Stock A has a beta of 0.50 and Stock B has a beta of 1.25. Suppose $r_f = 4\%$ and $\bar{R}_M = 10\%$. What is the expected return of an equally weighted portfolio of these two stocks?

Applying the SML, we have:

\begin{align*}
\bar{R}_A &= r_f + \beta_A(\bar{R}_M - r_f) = 4\% + 0.50(10\% - 4\%) = 7\%, \quad (93) \\
\bar{R}_B &= r_f + \beta_B(\bar{R}_M - r_f) = 4\% + 1.25(10\% - 4\%) = 11.5\% \quad (94)
\end{align*}

Expected return of an equally weighted portfolio is $0.50(7\%) + 0.50(11.5\%) = 9.25\%$.

**Example 14** Consider the data of the previous example. The portfolio’s beta is $0.50(0.50) + 0.50(1.25) = 0.875$. Applying the SML again, we have that

\[ \bar{R}_P = r_f + \beta_P(\bar{R}_M - r_f) = 4\% + (0.875)(10\% - 4\%) = 9.25\%. \]
15 Estimation of $\beta$ Via Regression

One can estimate $\beta$ via linear regression. Its value corresponds to the slope of the best-fitting line in the plot of the security’s excess return versus the market’s excess return. Linear regression models the excess return of a security as the sum of three components:

$$\hat{R}_i = \alpha_i + \beta_i \hat{R}_M + \varepsilon_i.$$  \hfill (95)

- $\alpha_i$ is the constant or intercept term of the regression, also called the stock’s alpha.
- $\beta_i \hat{R}_M$ represents the sensitivity of the stock to market risk.
- $\varepsilon_i$ is the error (or residual) term. The average error is assumed to be zero.

Taking the expectations of both sides of (95), we have

$$E[\hat{R}_i] = \beta_i E[\hat{R}_M] + \alpha_i.$$  \hfill (96)

The stock’s alpha, $\alpha_i$, measures the historical performance of the security relative to the expected return predicted by the SML. It is a risk-adjusted measure of a stock’s historical performance. According to the SML, $\alpha_i$ should not be significantly different from zero.

**Example 15** Suppose you estimate that stock A has a volatility of 26% and a beta of 1.45, whereas stock B has a volatility of 37% and a beta of 0.79. Suppose the risk-free rate is 3% and you estimate the market’s expected return as 8%.

- Which stock has more total risk? Which stock has more market risk?

  Total risk is measured by volatility; therefore stock B has higher total risk. Market risk is measured by beta; therefore stock A has higher market risk.

- Which firm has a higher cost of equity capital? The market premium is 8% - 3% = 5%.

  According to the SML:

  $$\bar{R}_A = 3\% + 1.45(5\%) = 10.25\%.$$  \hfill (97)
  $$\bar{R}_B = 3\% + 0.79(5\%) = 6.95\%.$$  \hfill (98)

  Market risk cannot be diversified. It is the market risk that determines the cost of capital. We conclude that firm A has the higher cost of equity capital even though it is less volatile.
There are three assumptions underlying the CAPM:

- Investors can buy or sell all securities at competitive market prices (without transaction costs) and can borrow and lend at the risk-free rate.
- Investors hold only efficient portfolios of traded assets. Such portfolios yield the maximum expected return for a given level of volatility.
- Investors have the same (homogeneous) expectations regarding the expected returns, volatilities and correlations of the securities.

These assumptions imply the following conclusions.

1. Each investor will identify the same portfolio of risky assets that has the highest Sharpe ratio. This is the tangent portfolio of risky assets.

2. The only difference between each investor is the amount of money invested in the risk-free asset, which depends on their respective appetites for risk. For example, one investor may choose to invest 50% of his wealth in the risk-free rate and the remaining 50% in the tangent portfolio. The actual weight of this investor’s portfolio in risky asset $i$ is simply $0.50x_i^*$. Another investor may choose to invest only 10% of her wealth in the risk-free asset; the actual weight of this investor’s portfolio in risky asset $i$ is $0.90x_i^*$.

3. The sum of all investor’s portfolios must equal the portfolio of all risky securities in the market, which is the market portfolio. Therefore, the tangent portfolio of risky securities equals the market portfolio. This statement can be viewed as saying that “supply equals demand.” As we alluded to before, prices in the market adjust so that all securities are held in the tangent (market) portfolio in just the right amounts determined by the SML.

4. According to the assumptions of CAPM, all mean-variance efficient portfolios must lie on the Capital Market Line (CML) defined by:

$$\bar{R}_P = r_f + \left( \frac{\bar{R}_M - r_f}{\sigma_M} \right) \sigma_P.$$

That is, a portfolio $P$ is efficient if and only if it has the same Sharpe ratio as the market’s, which is just another way of saying that all mean-variance efficient portfolios must be some combination of the risk-free asset and the market (tangent) portfolio.
17 Summary of key definitions, notation and formulae

Risk-free rate = $r_f$

Random return on asset $i$ = $R_i$

Expected return on asset $i$ = $E[R_i] = \bar{R}_i$

Expected excess return on asset $i$ = $\bar{R}_i - r_f$

Standard deviation of return on asset $i$ = $\sigma_i$

Covariance between returns on assets $i$ and $j$ = $\sigma_{ij}$

Covariance matrix = $\Sigma$

Correlation between returns on assets $i$ and $j$ = $\rho_{ij} = \sigma_{ij} / \sigma_i \sigma_j$

Portfolio weight on asset $i$ = $x_i$

Portfolio random return = $R_P = R(x) = \sum_i x_i R_i$

Portfolio expected return = $\sum_i x_i \bar{R}_i$

Portfolio expected excess return = $\sum_i x_i \hat{R}_i$

Portfolio variance $Var(R_P) = x^T \Sigma x = \sum_i \sum_j \sigma_{ij} x_i x_j$

Portfolio volatility $StDev(R_P) = \sqrt{Var(R_P)}$

Portfolio variance for two assets = $\sigma_i^2 x_i^2 + \sigma_j^2 x_j^2 + 2 x_i x_j \sigma_i \sigma_j \rho_{ij}$

Minimum variance portfolio of two assets = $\left( x_1^*, x_2^* \right) = \left( \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_{12}}, \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_{12}} \right)$

Variance of a large portfolio = $\frac{\text{AvgVar}}{n} + \left( 1 - \frac{1}{n} \right) * \text{AvgCov}$

Security $i$'s contribution to volatility of $P$ = $\frac{x_i \sigma_i}{\sqrt{x^T \Sigma x}} * \text{Corr}(R_i, R_P) * \text{Total risk of i} * \text{Fraction of i's risk common to P}$

Minimum variance portfolio = $x(\lambda) = \left( \frac{x_1(\lambda)}{x_2(\lambda)} \right) = \lambda \left( \Sigma^{-1} e \right)$,

Minimum variance = $\lambda = \frac{1}{e^T \Sigma^{-1} e} = \frac{1}{\text{sum of the entries of } \Sigma^{-1}}$

Sharpe ratio for a portfolio = $\mu(x) - r_f \sigma(x)^{-1} = \frac{R^T x - r_f^T x}{\sqrt{x^T \Sigma x}} = \frac{\text{Portfolio's expected excess return}}{\text{Portfolio's volatility}}$

Tangent portfolio = $x(\mu) = \left( \frac{x_1(\mu)}{x_2(\mu)} \right) = \mu \left( \Sigma^{-1} \hat{R} \right)$, $\mu = \frac{1}{e^T \Sigma^{-1} \hat{R}}$

Variance of Tangent Portfolio = $\mu * \text{Tangent portfolio's expected return}$

Security beta $\beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)} = \frac{\rho_{iM} \sigma_i \sigma_M}{\sigma_M^2} = \frac{\rho_{iM} \sigma_i}{\sigma_M}$

Security market line (SML): $\bar{R}_i = r_f + \beta_i (\bar{R}_M - r_f)$ = Required return on asset $i$

Capital market line (CML): $R_P = r_f + \left( \frac{\bar{R}_M - r_f}{\sigma_M} \right) \sigma_P$