Financial Options Analysis Homework Solutions

1. (a) State equations: $28h + 1.05m = 126$, $14h + 1.05m = 84 \implies h = 3$, $m = 40$.
   Cost of replicating portfolio $(3S, 40M) = 3(20) + 40 = 100$.
   
   (b) $q = \frac{20(1.05) - 14}{28 - 14} = 0.5$. $V_0 = \frac{E_p[V_1]}{1 + r_f} = \frac{0.5(126) + 0.5(84)}{1.05} = 100$.
   
   (c) $0.6(126) + 0.4(84) = 0.9226$.
   
   (d) $w_S = \frac{3(20)}{100} = 0.6$, $w_M = 0.4$. $E_p[r_S] = \frac{0.6(28) + 0.4(14)}{20} = 0.6(12) + 0.4(5) = 12\%$.
   $E_p[w_Sr_S + w_Mr_f] = 0.6(12) + 0.4(5) = 9.2\%$.
   
   (e) $RP_1$ for $e_1 = (1, 0)$ is $\left(\frac{1}{1.1}S, \frac{-1(1/14)(14)}{1.05}M\right)$. Cost of $RP_1 = 0.47619$.
   
   $RP_2$ for $e_2 = (0, 1)$ is $\left(\frac{1}{1.1}S, \frac{(1/14)(28)}{1.05}M\right)$. Cost of $RP_2 = 0.47619$.
   
   (f) $y_1 + y_2 = 0.95238 = 1/1.05$, as it should. $RP_1 + RP_2 = (0S, \frac{1}{1.05}M)$, as it should.

2. Payoffs for $V$ are $V_1(u) = 100$ and $V_1(d) = 50$.
   
   (a) State equations: $50h + 1.025m = 100$, $30h + 1.025m = 50 \implies h = 2.5$, $m = -24.39$.
   Cost of replicating portfolio $(3S, 40M) = 2.5(40) - 24.39 = 75.61$.
   
   (b) $q = \frac{40(1.025) - 30}{50 - 30} = 0.55$. $V_0 = \frac{E_p[V_1]}{1 + r_f} = \frac{0.55(1000 + 0.45(5))}{1.025} = 75.61$.
   
   (c) $0.7(100) + 0.3(50) = 1 = 12.42\%$.
   
   (d) $w_S = \frac{2.5(40)}{75.61} = 1.3226$, $w_M = -0.3226$. $E_p[r_S] = \frac{0.7(50) + 0.3(30)}{40} = 10\%$.
   $E_p[w_Sr_S + w_Mr_f] = 1.3226(10) - 0.3226(2.5) = 12.42\%$.
   
   (e) $RP_1$ for $e_1 = (1, 0)$ is $\left(\frac{1}{1.1}S, \frac{-1(1/20)(30)}{1.05}M\right)$. Cost of $RP_1 = 0.5366$.
   
   $RP_2$ for $e_2 = (0, 1)$ is $\left(\frac{1}{1.1}S, \frac{(1/20)(50)}{1.05}M\right)$. Cost of $RP_2 = 0.4390$.
   
   (f) $y_1 + y_2 = 0.9756 = 1/1.025$, as it should. $RP_1 + RP_2 = (0S, \frac{1}{1.025}M)$, as it should.

3. $q = \frac{1.1 - 0.80}{1.25 - 0.80} = 2/3$.
   
   (a) $V_2(ud) = V_2(du) = \max\{100 - 90, 0\} = 10$.
   
   $V_2(ud) = \max\{64 - 90, 0\} = 0$.
   
   $V_1(u) = \max\left\{\frac{2/3(66.25) + 1/3(10)}{1.1}, 125 - 90\right\} = 43.18$.
   
   $V_1(d) = \max\left\{\frac{2/3(10) + 1/3(0)}{1.1}, 80 - 90\right\} = 6.06$.
   
   $V_0 = \max\left\{\frac{2/3(43.18) + 1/3(6.06)}{1.1}, 100 - 100\right\} = 28.0073$.

(b) $V_0 = \frac{4/9(66.25) + 4/9(10) + 1/9(0)}{1.1} = 28.0073$.

NB: Values for European and American call options on a non-dividend paying stock will always be equal. In this setting, it will never be optimal to exercise early.

(c) $S_0 + P_0 - C_0 = PV(K)$. $100 + P_0 - 28.0073 = 90/1.1^2 \implies P_0 = 2.3875$. 

1
4. \( q = \frac{1.1^{0.80} - 0.80}{1.25^{0.80} - 0.80} = 2/3 \).

(a) 
\[
V_2(uu) = \max\{90 - 156.25, 0\} = 0.
\]
\[
V_2(ud) = V_2(du) = \max\{90 - 100, 0\} = 0.
\]
\[
V_2(dd) = \max\{90 - 64, 0\} = 26.
\]
\[
V_1(u) = \max\left\{ \frac{2/3(0) + 1/3(0)}{1.1}, 90 - 125 \right\} = 0.
\]
\[
V_1(d) = \max\left\{ \frac{2/3(0) + 1/3(26)}{1.1}, 90 - 80 \right\} = 10.
\]
\[
V_0 = \max\left\{ \frac{2/3(0) + 1/3(10)}{1.1}, 90 - 100 \right\} = 3.033.
\]

(b) \( V_0 = \frac{4/9(0.04) + 4/9(0.04) + 1/9(26)}{1.1^2} = 2.3875 \).

NB: Values for European and American put options on a non-dividend paying stock can be different. In this setting, it may be optimal to exercise early, as in state \( d \) at time \( t = 1 \).

(c) \( S_0 + P_0 - C_0 = PV(K). 100 + 2.3875 - C_0 = 90/1.1^2 \implies C_0 = 28.0073 \).

5. \( U = e^{0.60\sqrt{0.25}} = 1.3499 \). \( D = U^{-1} = 0.7408 \). \( q = \frac{e^{0.60\sqrt{0.25} - 0.7408}}{1.3499 - 0.7408} = 0.4358 \).

6. \( U = e^{0.80\sqrt{0.50}} = 1.7607 \). \( D = U^{-1} = 0.5680 \). \( q = \frac{e^{0.80\sqrt{0.50} - 0.5680}}{1.7607 - 0.5680} = 0.3791 \).

7. (a) \( p_1(u) = \frac{130(1.2) - 114}{181.5 - 114} = 0.622 \). \( p_1(d) = \frac{80(1.2) - 64}{114 - 64} = 0.64 \). \( p_0 = \frac{1.2(100) - 80}{130 - 80} = 0.80 \).

(b) Final payoffs are: \( V_2(uu) = 185 \). \( V_2(ud) = V_2(du) = 117.5 \). \( V_2(dd) = 167.5 \).
\[
V_0 = \frac{185(0.80)(0.622) + 117.5(0.80)(0.377) + 0.20(0.64) + 167.5(0.20)(0.36)}{1.1^2} = 107.43.
\]

(c) \( q_1(u) = \frac{130(1.04) - 114}{181.5 - 114} = 0.314 \). \( q_1(d) = \frac{80(1.04) - 64}{114 - 64} = 0.384 \). \( q_0 = \frac{1.04(100) - 80}{130 - 80} = 0.48 \).
\[
V_1(u) = \frac{0.314(185) + 0.686(117.5)}{1.04} = 133.36. \quad RP_1(u) = (1S, 3.36M).
\]
\[
V_1(d) = \frac{0.384(117.5) + 0.616(167.5)}{1.04} = 142.60. \quad RP_1(d) = (-1S, 222.60M).
\]
\[
V_0 = \frac{0.48(133.36) + 0.52(142.60)}{1.04} = 132.85. \quad RP_0 = (-0.185S, 151.3M).
\]

(d) Derivative security \( V \) would be underpriced, so you would want to buy it.

- At time \( t = 0 \):
  - You sell \( RP_0 \), that is, you borrow 151.3 and buy 0.185 units of stock.
  - You collect 132.85.
  - After buying one \( V \) for 107.43, you are left with 25.42.
  - Use the 25.42 to buy a very nice lunch at the student center (or do whatever you want with the money).

- At time \( t = 1 \), state = \( u \):
  - The value of the \( RP_0 \) you sold is now worth 133.36.
– Rebalance your portfolio: Sell $RP_1(u)$ for 133.36 and pay off your obligation.
– At time $t = 2$, you will owe either 185 or 117.5 on this rebalanced portfolio.
– Whichever state occurs, the $V$ you purchased will be worth exactly what you owe. It’s a “wash.”

• At time $t = 1$, state = $d$:
  – The value of the $RP_0$ you sold is now worth 142.60.
  – Rebalance your portfolio: Sell $RP_1(u)$ for 142.60 and pay off your obligation.
  – At time $t = 2$, the $V$ you purchased will be worth exactly what you owe. It’s a “wash.”

At each point in time you will always have what you need, so your 25.42 was safe and, indeed, you can spend it!

8. (a) $p_1(u) = \frac{150(1.12) - 119}{189 - 119} = 0.7$. $p_1(d) = \frac{70(1.12) - 49}{119 - 49} = 0.42$. $p_0 = \frac{1.12(100) - 70}{150 - 70} = 0.525$.
(b) Final payoffs are: $V_2(uu) = 8.36$. $V_2(ud) = V_2(du) = 0$. $V_2(dd) = 8.36$.
   $V_0 = \frac{8.36(0.525)(0.7) + (0.475)(0.58)}{1.12} = 4.289$.
(c) $q_1(u) = \frac{150(1.05) - 119}{189 - 119} = 0.55$. $q_1(d) = \frac{70(1.05) - 49}{119 - 49} = 0.35$. $q_0 = \frac{1.05(100) - 70}{150 - 70} = 0.4375$.

\[
\begin{align*}
V_1(u) &= \frac{0.55(8.36) + 0.45(0)}{1.05} = 4.3825. \\
R_P_1(u) &= (0.1195S, -13.5459M). \\
V_1(d) &= \frac{0.35(0) + 0.65(8.36)}{1.05} = 5.1793. \\
R_P_1(d) &= (-0.1195S, 13.5459M). \\
V_0 &= \frac{0.4375(4.3825) + 0.52625(5.1793)}{1.05} = 4.6007. \\
R_P_0 &= (-0.00996S, 5.5967M).
\end{align*}
\]

(d) Derivative security $V$ would be overpriced, so you would want to buy it.

• At time $t = 0$:
  – You sell $V$ for 5.
  – You buy the $RP_0$ for 4.6007, that is, you sell 0.00996 units of stock, and invest 5.5967 in the risk-free asset.
  – After selling one $V$ for 5 and buying $RP_0$, you are left with 0.3993.
• At time $t = 1$, state = $u$:
  – The value of the $RP_0$ you bought is now worth 4.3825.
  – Rebalance your portfolio: Sell $RP_0$ for 4.3825 and buy $RP_1(u)$.
  – At time $t = 2$, you will owe either 8.36 or 0 on the $V$ you sold.
  – Whichever state occurs, the $RP_1(u)$ will be worth exactly what you owe. It’s a “wash.”
• At time $t = 1$, state = $d$:
  – The value of the $RP_0$ you bought is now worth 5.1793.
  – Rebalance your portfolio: Sell $RP_0$ for 4.3825 and buy $RP_1(d)$.
  – At time $t = 2$, you will owe either 0 or 8.36 on the $V$ you sold.
  – Whichever state occurs, the $RP_1(d)$ will be worth exactly what you owe. It’s a “wash.”

At each point in time you will always have what you need, so your 0.3993 was safe!
9. (a) Price each asset: $26y_1 + 16y_2 = 20, 10y_1 + 10y_2 = 10 \implies y_1 = 10/21, y_2 = 10/21.
   
   (b) $80 = 68y_1 + Zy_2 \implies Z = 100.

   (c) $1 + r_f = (y_1 + y_2)^{-1} \implies r_f = 5%.$

10. (a) Since $S_2$ does not pay off in either states 1 or 2, only $S_1$ and $S_3$ can be used to replicate the payoffs of $V$ in these two states.
    
    Since $S_3$ is our old friend $M$, we’re back to “(hS1, S3)”: $h = \frac{420-460}{200-240} = 1$ and $S_3 = 200$.
    
    Now use state 3 to pin down the number of units of $S_2$ to hold:
    
    $176S_2 + 1.1(200)44 \implies S_2 = -1.$
    
    Thus, the $RP = (1S_1, -1S_2, 200S3)$ for a cost of 220.

   (b) Let $q = (q_1, q_2, q_3)$ denote the risk-neutral probability vector. Obviously, $r_f = 10%$.
    
    Risk-neutral valuation of $S_2$ implies that $q_3 = 0.5$. Thus, $q_1 + q_2 = 0.5$.
    
    Risk-neutral valuation of $S_1$ implies that $100 = \frac{200q_1+240q_2}{1.1}$. Thus, $q_1 = q_2 = 0.25$.
    
    Risk-neutral valuation of $V$ is $\frac{0.25(220)+0.25(460)+0.5(44)}{1.1} = 220$, as it should.

11. First four securities show that $q = 0.5$.
    
    To be consistent, the value of $S_9 = \frac{0.5(385)+0.5(245)}{1.05} = 300$, which is less than 305.
    
    The portfolio $(10S_1, 100S_2)$ replicates $S_9$ and costs 300.
    
    So, sell $S_9$ for 305, buy this RP for 300, pocket the difference of 5.

12. (a) We can use the first four securities to determine the state-prices:
    
    $125y_1 + 80y_2 = 60$. 
    
    $110y_1 + 110y_2 = 60$. \implies $y_1 = 4/11, y_2 = 2/11$.
    
    $28y_3 + 14y_4 = 8$. 
    
    $105y_3 + 105y_4 = 40$. \implies $y_3 = 4/21, y_4 = 4/21$.
    
    No-arbitrage value of security 5 is $(4/11)22 + (2/11)11 + (4/12)21 + (4/21)42 = 22$.
    
    Since this IS the price of security 5, this market does NOT exhibit arbitrage.

    (b) $1 + r_f = (y_1 + y_2 + y_3 + y_4)^{-1} = 214/231 \implies r_f = 7.94%$.
    
    Alternatively, a purchase of 1 unit of security 3 and $110/105$ units of security 4 yields a constant payoff of 110. The cost of this portfolio is $60(1) + 40(110/105) = 101.905$. So, the riskless return on this portfolio is $110/101.905 - 1 = 7.94\%$, as it should.

13. $q_1(u) = \frac{125(1.025)-100}{1.5625-100} = 0.5$. $q_1(d) = q_1(u) = 0.5$. $q_0 = \frac{1.1(100)-80}{125-80} = 2/3$.
    
   (a) $V_2(uu) = 0, V_2(ud) = 25, V_2(du) = 0, V_2(dd) = 36$.
    
   $V_0 = \frac{(2/3)(0.5)25+(1/3)(0.5)36}{1.1(1.025)} = 12.71$.

   (b) $V_2(uu) = 56.25, V_2(ud) = 0, V_2(du) = 20, V_2(dd) = 0$.
    
   $V_0 = \frac{(2/3)(56.25)+(1/3)(0.5)20}{1.1(1.025)} = 19.586$.

14. $q_1(u) = \frac{130(1.05)-102}{171-102} = 0.5$. $q_1(d) = \frac{80(1.05)-57}{102-57} = 0.6$. $q_0 = \frac{100(1.05)-80}{130-80} = 0.5$.
    
   $Y_2(uu)/3 = 133.6, Y_2(ud)/3 = 110.6, Y_2(du)/3 = 94, Y_2(dd)/3 = 79$.
    
   $V_0 = \frac{(0.5)(0.6)10+0.5(0.4)25}{1.05^2} = 7.2562$.

15. $q_0 = 0.43$. $q_1(u) = q_1(d) = 0.52$. 


Table 1: \(S\) and \(V\) payoffs for problem 15a.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(S)</th>
<th>(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(S_0 = 36)</td>
<td>(V_0 = \frac{0.43(21.27)+0.57(5.6727)}{1.925} = 12.0787)</td>
</tr>
<tr>
<td>1</td>
<td>(S_1(u) = 54)</td>
<td>(V_1(u) = \max{\frac{0.52(45)+0.48(0)}{1.1}, 54 - 36} = 21.27)</td>
</tr>
<tr>
<td>2</td>
<td>(S_2(uu) = 81)</td>
<td>(S_2(du) = 36)</td>
</tr>
</tbody>
</table>

Table 2: \(S\) and \(V\) payoffs for problem 15b.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(S)</th>
<th>(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(S_0 = 36)</td>
<td>(V_0 = \frac{0.43(7.8545)+0.57(12)}{1.025} = 12.0787)</td>
</tr>
<tr>
<td>1</td>
<td>(S_1(u) = 54)</td>
<td>(V_1(u) = \max{\frac{0.52(0)+0.48(18)}{1.1}, 54 - 54} = 7.8545)</td>
</tr>
<tr>
<td>2</td>
<td>(S_2(uu) = 81)</td>
<td>(S_2(du) = 36)</td>
</tr>
</tbody>
</table>

\(V_1(d) = \max\{\frac{0.52(0)+0.48(18)}{1.1}, 36 - 24\} = 12\)
\(V_2(du) = 0\)
\(S_2(dd) = 16\)
\(V_2(dd) = 20\)