A Verification Based Method to Generate Cutting Planes for IPs

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SIAM Conference on Optimization, 2011
Outline

Cutting Plane Operators

Generating Cutting Plane Differently: Design and Verify

Verification Closure and Basic Properties

Ranks using Verification Cuts
   Upper Bound on Ranks using Verification Cuts
   Lower Bound on Ranks using Verification Cuts
1 Cutting Plane Operators
A basic question in integer programming

Question:
Given a closed convex set \( P \subseteq \mathbb{R}^n \), obtain

\[
\text{Convex hull of } (P \cap \mathbb{Z}^n) =: P_I
\]
A basic question in integer programming

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*Given a closed convex set* $P \subseteq \mathbb{R}^n$, *obtain*  

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1. This is a difficult task.
A basic question in integer programming

Question:
Given a closed convex set $P \subseteq \mathbb{R}^n$, obtain

$$\text{Convex hull of } (P \cap \mathbb{Z}^n) =: P_I$$

1. This is a difficult task.
2. Typically we are happy to obtain relaxation of the convex hull of $P \cap \mathbb{Z}^n$, i.e.,

$$T \subseteq \mathbb{R}^n \text{ s.t. } P \supseteq T \supseteq P_I. \quad (1)$$
A basic question in integer programming

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   \[
   T \subseteq \mathbb{R}^n \text{ s.t. } P \supseteq T \supseteq P_I. 
   \]

3. Let \( M \) be an \textit{cutting plane operator} applied to \( P \) to obtain a (closed) convex relaxation of \( (P \cap \mathbb{Z}^n) \), i.e.
   \[
   P \supseteq M(P) \supseteq P_I. 
   \]
Cutting planes: the way we use these operators

- If $\langle a, x \rangle \leq b$ is a valid inequality for $M(P)$, then it is a valid inequality for $(P \cap \mathbb{Z}^n)$. 
Examples of some well-known cutting-plane operators

- In this presentation, \( P \subseteq \mathbb{R}^n \) is always a rational polytope.
- Some operators for general integer programs:
  1. Gomory-Chvátal closure (GC)
  2. Split disjunctive closure (SC)
- Some operators for 0 – 1 integer programs, i.e, \( P \subseteq [0, 1]^n \) (Lovász-Schrijver operators):
  1. Lift-and-project operator \( (N_0) \)
  2. \( N \)
  3. \( N_+ \)
Admissible cutting-plane procedures for 0-1 polytopes

All known ‘reasonable’ cutting plane operators satisfy the following properties:

**Definition**

A cutting-plane procedure $M$ defined for a rational polytope $P := \{x \in [0, 1]^n \mid Ax \leq b\}$ is **admissible** if the following holds:

1. **Validity:** $P_I \subseteq M(P) \subseteq P$.
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A cutting-plane procedure $M$ defined for a rational polytope $P := \{x \in [0, 1]^n \mid Ax \leq b\}$ is admissible if the following holds:

1. **Validity**: $P_I \subseteq M(P) \subseteq P$.
2. **Inclusion Preservation**: If $P \subseteq Q$, then $M(P) \subseteq M(Q)$ for all polytopes $P, Q \subseteq [0, 1]^n$.

Definition is modified from [Pokutta and Schulz (2009)]
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3. Homogeneity: $M(F \cap P) = F \cap M(P)$, for all faces $F$ of $[0, 1]^n$. 

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4. **Substitution Independence:** Let $\varphi_F$ be the projection onto the face $F$ of $[0, 1]^n$. Then $\varphi_F(M(P \cap F)) = M(\varphi_F(P \cap F))$. 

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5. **Single Coordinate Rounding:** If $x_i \leq \epsilon < 1$ (or $x_i \geq \epsilon > 0$) is valid for $P$, then $x_i \leq 0$ (or $x_i \geq 1$) is valid for $M(P)$. 

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6. **Commuting with Coordinate Flips and Duplications**: $\tau_i(M(P)) = M(\tau_i(P))$, where $\tau_i$ is either one of the following two operations: (i) Coordinate flip: $\tau_i : [0, 1]^n \to [0, 1]^n$ with $(\tau_i(x))_i = (1 - x_i)$ and $(\tau_i(x))_j = x_j$ for $j \in [n] \setminus \{i\}$; (ii) Coordinate Duplication: $\tau_i : [0, 1]^n \to [0, 1]^{n+1}$ with $(\tau_i(x))_{n+1} = x_i$ and $(\tau_i(x))_j = x_j$ for $j \in [n]$.
Admissible cutting-plane procedures for 0-1 polytopes

All known ‘reasonable’ cutting plane operators satisfy the following properties:

Definition
A cutting-plane procedure $M$ defined for a rational polytope $P := \{ x \in [0, 1]^n \mid Ax \leq b \}$ is admissible if the following holds:

1. **Validity:** $P_I \subseteq M(P) \subseteq P$.
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7. **Short Verification:** There exists a polynomial $p$ such that for any inequality $\langle c, x \rangle \leq d$ that is valid for $M(P)$ there is a set $I \subseteq [m]$ with $|I| \leq p(n)$ such that $\langle c, x \rangle \leq d$ is valid for $M(\{ x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq b_i, i \in I \})$.

Definition is modified from [Pokutta and Schulz (2009)]
Admissible cutting-plane procedures for general polytopes

Definition
If $M$ is defined for general rational polytopes $P \subseteq \mathbb{R}^n$, then we say $M$ is admissible if
- $M$ satisfies (1.)-(7.) when restricted to polytopes contained in $[0, 1]^n$ and
- for general polytopes $P \subseteq \mathbb{R}^n$, $M$ satisfies VALIDITY, INCLUSION PRESERVATION, SHORT VERIFICATION and HOMOGENEITY is replaced by
  8. **Strong Homogeneity:** If $P \subseteq F^\leq := \{x \in \mathbb{R}^n \mid \langle a, x \rangle \leq b\}$ and $F = \{x \in \mathbb{R}^n \mid \langle a, x \rangle = b\}$ where $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}$, then $M(F \cap P) = M(P) \cap F$.

Definition
If $M$ satisfies all required properties for being admissible except SHORT VERIFICATION, then we say $M$ is *almost admissible*.

Definition is modified from [Pokutta and Schulz (2009)]
2 Generating Cutting Plane Differently: Design and Verify
The Computation Scheme

**Input:**
\[ P := \{ x \in \mathbb{R}^n \mid Ax \leq b \} \]

→ Black box operator M

**Output:**
\[ \langle c, x \rangle \leq d \]
valid for \( M(P) \)
Computation vs. Verification Scheme

Computation Scheme

\[ P := \{ x \in \mathbb{R}^n \mid Ax \leq b \} \rightarrow \text{Black box operator } M \rightarrow \langle c, x \rangle \leq d \text{ valid for } M(P) \]
Computation vs. Verification Scheme

**Computation Scheme**

\[
\begin{align*}
\text{Input :} & \quad P := \{x \in \mathbb{R}^n \mid Ax \leq b\} \\
\rightarrow & \quad \text{Black box operator } M \\
\text{Output :} & \quad \langle c, x \rangle \leq d
\end{align*}
\]

valid for \( M(P) \)

**Verification Scheme**

\[
\begin{align*}
\text{Input :} & \quad P := \{x \in \mathbb{R}^n \mid Ax \leq b\} \\
\rightarrow & \quad \text{Unfaithful Oracle to generate cuts} \\
\text{Inequality claimed to be valid :} & \quad \langle c, x \rangle \leq d
\end{align*}
\]

↓

Use Black Box operator to verify validity

↓

If valid, then Output : \( \langle c, x \rangle \leq d \)

A similar idea discussed in Cook, Coullard, Turán (1987).
How to verify validity of cut using black box operator

Assuming $c \in \mathbb{Z}^n$, $d \in \mathbb{Z}$:

\[
\langle c, x \rangle \leq d \text{ is valid for } P_I
\]

\[
\vdash
\]

\[
(P \cap \{x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1\})_I = \emptyset
\]
How to verify validity of cut using black box operator

Assuming $c \in \mathbb{Z}^n$, $d \in \mathbb{Z}$:

$$\langle c, x \rangle \leq d$$

is valid for $P_l$

$$\uparrow$$

$$M ( P \cap \{ x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1 \}) = \emptyset$$

So if $M ( P \cap \{ x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1 \}) = \emptyset$, then we declare $\langle c, x \rangle \leq d$ is a valid inequality.
Question:

How much do we gain (if at all?) from having to only verify that a given inequality is valid for $P_I$, rather than actually computing it.
Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for $P_I$:

$\quad M(P) \cap \{ x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1 \} = \emptyset$ (computation)

$\quad M(P \cap \{ x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1 \}) \neq \emptyset$ (verification)

The strength of the verification scheme lies in the following inclusion that is satisfied by every "reasonable" cutting plane operator.

$M(P \cap \{ x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1 \}) \subseteq M(P) \cap \{ x \in \mathbb{R}^n | \langle c, x \rangle \geq d + 1 \}$. 

This inclusion can be strict!

1. That is, suppose there exists $(c_0, d_0) \in \mathbb{Z}^n_{++}$ such that $M(P \cap \{ x \in \mathbb{R}^n | \langle c_0, x \rangle \geq d_0 + 1 \}) = \emptyset$ and (3) $M(P) \cap \{ x \in \mathbb{R}^n | \langle c_0, x \rangle \geq d_0 + 1 \} \neq \emptyset$ (4)

2. Then $\langle c_0, x \rangle \leq d_0$ is verifiable but not computable.
Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for $P_I$:

(i) $M(P) \cap \{x \mid \langle c, x \rangle \geq d + 1\} = \emptyset$  

(computation)
Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for $P_I$:

(i) $M(P) \cap \{x \mid \langle c, x \rangle \geq d + 1\} = \emptyset$  \hspace{1cm} (computation)

(ii) $M(P \cap \{x \mid \langle c, x \rangle \geq d + 1\}) = \emptyset$  \hspace{1cm} (verification)
Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for $P_I$:

(i) $M(P) \cap \{x | \langle c, x \rangle \geq d + 1\} = \emptyset$ \hspace{1cm} (computation)

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Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for $P_I$:

(i) $M(P) \cap \{ x \mid \langle c, x \rangle \geq d + 1 \} = \emptyset$ (computation)

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The strength of the verification scheme lies in the following inclusion that is satisfied by every "reasonable" cutting plane operator.

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This inclusion can be strict!

1. That is, suppose there exists $(c^0, d^0) \in \mathbb{Z}^{n+1}$ such that

$$M(P \cap \{ x \mid \langle c^0, x \rangle \geq d^0 + 1 \}) = \emptyset$$  and \hspace{1cm} (3)

$$M(P) \cap \{ x \mid \langle c^0, x \rangle \geq d^0 + 1 \} \neq \emptyset$$  \hspace{1cm} (4)

2. Then $\langle c^0, x \rangle \leq d^0$ is verifiable but not computable.
3 Verification Closure and Basic Properties
Verification Closure: Definition and Basic Property

Verification Closure

Definition
Let $M$ be admissible. Then

$$\partial M(P) := \bigcap_{(c,d) \in \mathbb{Z}^{n+1}} \{ \langle c, x \rangle \leq d \mid M(P \cap \{ \langle c, x \rangle \geq d + 1 \}) = \emptyset \}$$

is the verification scheme closure of $M$.

We add all inequalities whose validity can be verified with application of $M$. 
Verification Closure

Definition
Let $M$ be admissible. Then

$$\partial M(P) := \bigcap_{(c,d) \in \mathbb{Z}^{n+1}} \{ \langle c, x \rangle \leq d \mid M(P \cap \{ \langle c, x \rangle \geq d + 1 \}) = \emptyset \}$$

is the verification scheme closure of $M$.

We add all inequalities whose validity can be verified with application of $M$.

Theorem
If $M$ is admissible, then $\partial M$ is almost admissible.
Comparing closures

Properties of general admissible $M$

Theorem

1. $\partial M(P) \subseteq M(P)$ for all rational polytopes $P$. There exists a rational polytope $Q$, such that $\partial M(Q) \subsetneq M(Q)$.

2. $\partial M(P) \subseteq GC(P) \cap N_0(P)$ for all rational polytopes $P$. There exists a rational polytope $Q$, such that $\partial M(Q) \subsetneq GC(Q) \cap N_0(Q)$. 
Comparing closures

Properties of general admissible $M$

Theorem

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Comparing specific closures

Theorem

Let $L$ and $M$ be admissible cutting plane operators such that $L(P) \subseteq M(P)$ for all rational polytopes $P$. Then $\partial L(P) \subseteq \partial M(P)$ for all rational polytopes $P$. Moreover,

1. $\partial GC(P) \subseteq SC(P)$ for all rational polytopes $P$. There exists a rational polytope $Q$, such that $\partial GC(Q) \subset SC(Q)$.

2. There exist polytopes $Q^1$ and $Q^2$, such that $\partial N_0(Q^1) \subset \partial GC(Q^1)$ and $\partial N_0(Q^2) \supset \partial GC(Q^2)$.

3. There exist polytopes $Q^1$ and $Q^2$, such that $\partial N_0(Q^1) \subset SC(Q^1)$ and $\partial N_0(Q^2) \supset SC(Q^2)$.

4. There exists a rational polytope $Q$, such that $\partial N(Q) \subset \partial N_0(Q)$.
Comparing closures
4.1
Upper Bound on Ranks using Verification Cuts
\[ \partial GC \text{ in } \mathbb{R}^2 \]

**Theorem**
\[ \partial GC(P) = P_1 \text{ for all rational polytopes } P \subseteq \mathbb{R}^2, \text{ that is } \text{rk}_{\partial M}(P) = 1 \text{ for all rational polytopes } P \subseteq \mathbb{R}^2. \]
"Mimicking" the $N_+$ operator

**Lemma**
Let $M$ be admissible and $P \subseteq [0, 1]^n$. Further let $(c, d) \in \mathbb{Z}_+^{n+1}$. If $\langle c, x \rangle \leq d$ is valid for $P \cap \{x \mid x_i = 1\}$ for every $i \in [n]$ with $c_i > 0$, then $\langle c, x \rangle \leq d$ is valid for $\partial M(P)$. 
"Mimicking" the $N_+$ operator

**Lemma**
Let $M$ be admissible and $P \subseteq [0, 1]^n$. Further let $(c, d) \in \mathbb{Z}_{+}^{n+1}$. If $\langle c, x \rangle \leq d$ is valid for $P \cap \{x \mid x_i = 1\}$ for every $i \in [n]$ with $c_i > 0$, then $\langle c, x \rangle \leq d$ is valid for $\partial M(P)$.

**Stable Set Polytope**
Given a graph $G := (V, E)$, let $FSTAB(G) := \{x \in [0, 1]^n \mid x_u + x_v \leq 1 \ \forall (u, v) \in E\}$.

**Theorem**
Clique Inequalities, odd hole inequalities, odd anti-hole inequalities, and odd wheel inequalities are valid for $\partial M(FSTAB(G))$ with $M$ being an admissible operator.

Proof uses previous lemma and ideas from [Lovász Schrijver (1991)].
"Mimicking" the $N_+$ operator

**Lemma**

Let $M$ be admissible and $P \subseteq [0, 1]^n$. Further let $(c, d) \in \mathbb{Z}_+^{n+1}$. If $\langle c, x \rangle \leq d$ is valid for $P \cap \{ x \mid x_i = 1 \}$ for every $i \in [n]$ with $c_i > 0$, then $\langle c, x \rangle \leq d$ is valid for $\partial M(P)$.

**Monotone Polytope**

We say $P \subseteq [0, 1]^n$ is monotone (or of anti-blocking type) if $y \in P$ whenever $y \leq x$ and $x \in P$.

**Theorem**

Let $M$ be admissible and $P \subseteq [0, 1]^n$ be a monotone polytope with $\max_{x \in P} e_x = k$. Then $rk_{\partial M}(P) \leq k + 1$.

Proof uses previous lemma and ideas from [Cook Dash (2001)].
4.2 Lower Bound on Ranks using Verification Cuts
Not the paragon of perfection: lower bound on rank of $A_n$

$$A_n := \left\{ x \in [0, 1]^n \mid \sum_{i \in I} x_i + \sum_{i \not\in I} (1 - x_i) \geq \frac{1}{2} \quad \forall I \subseteq [n] \right\}.$$ 

**Lemma**

Let $M$ be admissible and let $\ell \in \mathbb{N}$ such that $rk_M(A_n) \geq \left\lfloor \frac{n}{\ell} \right\rfloor$. If $n > 2\ell + 1$, then

$$rk_{\partial M}(A_n) \geq \left\lfloor \frac{n - 1}{2\ell + 1} \right\rfloor.$$
Not the paragon of perfection: lower bound on rank of $A_n$

\[ A_n := \left\{ x \in [0, 1]^n \mid \sum_{i \in I} x_i + \sum_{i \not\in I} (1 - x_i) \geq \frac{1}{2} \quad \forall I \subseteq [n] \right\}. \]

**Lemma**

Let $M$ be admissible and let $\ell \in \mathbb{N}$ such that $rk_M(A_n) \geq \left\lfloor \frac{n}{\ell} \right\rfloor$. If $n > 2\ell + 1$, then

\[ rk_{\partial M}(A_n) \geq \left\lceil \frac{n - 1}{2\ell + 1} \right\rceil. \]

**Corollary**

Let $M \in \{GC, N_0, N, N_+, SC\}$ and $n \in \mathbb{N}$ with $n \geq 4$. Then $rk_{\partial M}(A_n) \geq \left\lceil \frac{n-1}{3} \right\rceil$. 
Not the paragon of perfection: lower bound on rank of $A_n$

**Corollary**
Let $M \in \{GC, N_0, N, N_+, SC\}$ and $n \in \mathbb{N}$ with $n \geq 4$. Then $rk_{\partial M}(A_n) \geq \left\lfloor \frac{n-1}{3} \right\rfloor$.

**Traveling Salesman Polytope**
Let $G = (V, E)$ be the undirected complete graph on $n$ vertices. The *subtour elimination polytope* $H_n \subseteq [0, 1]^{|E|}$ is the set
\[
\begin{align*}
    x(\delta(\{v\})) &= 2 & \forall \ v \in V \\
    x(E(W)) &\leq |W| - 1 & \forall \emptyset \subsetneq W \subsetneq V \\
    x_e &\in [0, 1] & \forall e \in E.
\end{align*}
\]

**Theorem**
Let $M \in \{GC, SC, N_0, N, N_+\}$. For $n \in \mathbb{N}$ and $H_n$ as defined above we have
\[
\left\lfloor \frac{\lfloor n/8 \rfloor - 1}{3} \right\rfloor \leq rk_{\partial M}(H_n) \leq n + 1.
\]

Proof uses results from [Chvátal, Cook, Hartmann (1989)] [Cook Dash (2001)].