Facets for Two-Dimensional Mixed Integer Infinite Group Problem

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Motivation

- Generation of **strong cutting planes** will help solve general MIPs faster.
- All know group based cutting planes using a single constraint to derive cut.
- We present first know strong cuts that use **two constraints**.
- We hope these will be stronger since they use information from two constraints concurrently.
Outline

1. Infinite Group Relaxation
   - Introduction

2. Tools to Prove Facets
   - Subadditivity
   - Interval Lemma
   - Homomorphism

3. Facet-defining Inequalities
   - Family - I
   - Family - II

4. Conclusion
Standard IP:

\[ Ax = b \quad x \in \mathbb{Z}_+ , \]

where \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^{m \times 1} \).

Relaxation step 1: Consider each row modulo 1.

\[ \sum_{i=1}^{n} (A_{ij}(mod 1)) x_i \equiv b_j (mod 1) \quad \forall 1 \leq j \leq m \tag{1} \]

Rewrite in Group Space: \( \sum_{i=1}^{n} (a_i) x_i = r \)

Each \( a_i \) belongs to the group \( I^m = \{ x \in \mathbb{R}^m | 0 \leq x_i < 1 \quad \forall 1 \leq i \leq m \} \).

Note that \( a_i = (A_{i1}(mod 1), \ldots, A_{im}(mod 1)) \).

Relaxation step 2: Introduce new variables.

\[ \sum_{a \in I^m} ax(a) = r \tag{2} \]
Definition: Group Problem and Valid Inequalities

**Definition (Integer Group Problem \(PI(r, m)\), Johnson 1974)**

For \(r \in I^m\) and \(r \neq o\), the group problem \(PI(r, m)\) is the set of functions \(t : I^m \rightarrow \mathbb{R}\) such that

1. \(\sum_{u \in I^m} ut(u) = r, \ r \in I^m\),
2. \(t(u)\) is a non-negative integer for \(u \in I^m\),
3. \(t\) has a finite support, i.e., \(t(u) > 0\) for a finite subset of \(I^m\).

**Definition (Valid Inequality, Johnson 1974)**

A function \(\phi : I^m \rightarrow \mathbb{R}_+\) is defined as a valid inequality for \(PI(r, m)\) if \(\phi(o) = 0, \ \phi(r) = 1\) and \(\sum_{u \in I^m} \phi(u)t(u) \geq 1, \ \forall \ t \in PI(r, m)\).
**Piecewise Linear Functions**

**Definition**

$\phi$ is piecewise linear, i.e. $l^2$ can be decomposed into finitely many polytopes with non-empty interiors $P_1, \ldots, P_k$, such that

$$\phi(u) = \alpha_t^T u + \beta_t, \ \forall u \in P_t,$$

where $\alpha_t \in \mathbb{R}^{2 \times 1}$, $\beta_t \in \mathbb{R}$ $\forall t = \{1, 2, \ldots, k\}$. 
Step to Prove Function Represents Facet-defining Inequality

1. Prove function is subadditive, i.e., $\phi(u) + \phi(v) \geq \phi(u + v)$
   $\forall u, v \in I^2$.
   - Develop methods to efficiently prove a function is subadditive over $I^2$.

2. Prove function is minimal (un-dominated). This is easy to do.
   (Gomory and Johnson Theorem 1972).

3. Define an additive equality as $\phi(u) + \phi(v) = \phi(u + v)$. Let $E(\phi)$ be the set of all additive equalities. Then if the $\phi$ is the only function that satisfies all the equalities $E(\phi)$, then $\phi$ is a facet.
   - Prove a result called Interval Lemma in two dimension. This result is used to prove $E(\phi)$ is unique.
   - Prove a homomorphism result to generate new facets from older ones.
Checking subadditivity for Functions Defined on $l^2$

**Theorem (Checking Subadditivity)**

Let $\phi$ be a continuous, piecewise linear and nonnegative function on $l^2$. Then $\phi$ is subadditive iff

$$\phi(v_1) + \phi(v_2) \geq \phi(v_1 + v_2) \quad \forall v_1, v_2 \in V(\phi) \cup V'(\phi)$$  \hspace{1cm} (3)

$$\phi(v_1) + \phi(v_3 - v_1) \geq \phi(v_3) \quad \forall v_1, v_3 \in V(\phi) \cup V'(\phi)$$  \hspace{1cm} (4)

$$\phi(v_1) + \phi(e_2) \geq \phi(e_3) \quad \text{where } e_2 \in q_2, e_3 \in q_3, v_1 + e_2 = e_3,$$

$$\forall v_1 \in V(\phi) \cup V'(\phi), \quad \forall q_2, q_3 \in Q(\phi)  \hspace{1cm} (5)$$

$$\phi(e_1) + \phi(e_2) \geq \phi(v_3) \quad \text{where } e_1 \in q_1, e_2 \in q_2, e_1 + e_2 = v_3,$$

$$\forall v_3 \in V(\phi) \cup V'(\phi), \quad \forall q_1, q_2 \in Q(\phi)  \hspace{1cm} (6)$$

Furthermore, if $e_2$ and $e_3$ (resp. $e_1$ and $e_2$) belong to identical or parallel edges, then (5) (resp. (6)) is redundant.
Discussion on Subadditivity Result
Interval lemma in Two Dimensions

- Interval Lemma is a key tool used to prove that a function is facet-defining by Gomory and Johnson [2003].
- The following a generalization we introduced in two dimensions.

**Theorem (Interval Lemma in Two Dimensions)**

Let $U$ and $V$ be closed sets in $\mathbb{R}^2$. Let $g$ be a real-valued function defined over $U$, $V$ and $U + V$. Assume that

1. $U$ is star-shaped with respect to the origin, and $U$ has a non-empty interior.
2. $V$ is path connected.
3. $g(u) + g(v) = g(u + v), \forall u \in U, \forall v \in V$.
4. $\sum_{i \in S} g(u_i) = g(\sum_{i \in S} u_i) \forall u_i \in U$ such that $\sum_{i \in S} u_i \in U$ and $\forall S$ with $|S| \leq 3$.
5. $g(u) \geq 0, \forall u \in U$.

Then $g$ is a linear function with the same gradient in $U$, $V$ and $U + V$. 
Creating New Facets

**Definition (λ Homomorphism)**

The homomorphism $\lambda : I^2 \to I^2$ is defined as $\lambda(x, y) = (\lambda_1 x (mod 1), \lambda_2 y (mod 1))$, where $\lambda_1, \lambda_2$ are positive integers.

**Theorem (Homomorphism Theorem)**

$\phi$ is facet-defining with respect to right-hand-side $r$ iff $\phi \circ \lambda$ is facet-defining with respect to right-hand-side $v$, where $\lambda(v) = r$. 
Family 1: Constraint Aggregation

Construction

Given $\zeta$ a piecewise linear and continuous valid inequality for one dimensional integer infinite group problem $PI(c, 1)$, we construct the function $\tau$ for $PI(r, 2)$ with right-hand-side $r \equiv (f_1, f_2)$ where $\lambda_1 f_1 + \lambda_2 f_2 = c$ as $\tau(x, y) = \zeta(\lambda_1 x + \lambda_2 y)(mod 1)$, and $\lambda_1, \lambda_2 \in \mathbb{Z}$ and are not both zero.

Theorem (Aggregation Theorem)

$\tau$ is facet-defining for $PI(r, 2)$ iff $\zeta$ is facet-defining for 1DIIIGP.
Theorem (Two Gradient Theorem)

Any continuous piecewise linear two-gradient facet of Pl(r,2) can be derived from a facet of Pl(r',1) using Construction 1.

Some observations:

➢ Gives a complete characterization of continuous functions with only two gradients.

➢ All two slope functions for 1DIIGP are facet-defining [Gomory and Johnson 1972, 2003]. This is a two-dimensional analog for a similar result for the one dimensional infinite group problem.
Family 2: Three-Gradient Facet

Construction

We divide $l^2$ into five polytopes $R_1, R_2, R_3, R_4, R_5$ as shown in figure.

We construct $\psi$ to be the only continuous piecewise linear function with $\psi(f_1, f_2) = 1$ and $\psi(0, 0) = 0$, whose gradients in $R_2$ and $R_4$ are equal and whose gradients in $R_3$ and $R_5$ are equal.

Theorem

$\psi$ is facet-defining for $PI(r, 2)$. 
Three-Gradient Functions Yield Facets of IPs

Example

Consider the set of nonnegative integer solutions to

\[
\begin{bmatrix}
8 \\
0
\end{bmatrix}x_1 + \begin{bmatrix}
2 \\
0
\end{bmatrix}x_2 + \begin{bmatrix}
1 \\
7
\end{bmatrix}x_3 + \begin{bmatrix}
5 \\
2
\end{bmatrix}x_4 + \begin{bmatrix}
6 \\
3
\end{bmatrix}x_5 + \begin{bmatrix}
4 \\
1
\end{bmatrix}x_6 + \begin{bmatrix}
0 \\
8
\end{bmatrix}x_7 = \begin{bmatrix}
12 \\
12
\end{bmatrix}
\]

where \( x_i \in \mathbb{Z}_+ \ \forall i \in \{1, \ldots, 7\} \). This system has 3 feasible solutions:

\{0 0 1 1 1 0 0\}, \{0 1 0 2 0 0 1\} and

\{0 1 0 0 1 1 1\}. Now consider the constraints divided by 8.

Observations

1. The three-gradient inequality \( \psi \) is generates a facet of the feasible region of the IP.
2. The GMIC generates a different facet of this problem.
Using result from Johnson [1974] cuts for integer infinite groups can be extended to mixed integer infinite group problems.

**Proposition**

Among all the facets of one-dimensional mixed integer infinite group problem (1DMIIGP), the coefficients of continuous variables are strongest in GMIC.

**Proposition**

The coefficients for continuous variables of the three-gradient inequality $\psi$ are not dominated by GMIC based on the single constraint.

**Figure:** Three-gradient not dominated by GMIC
Presented Tools for proving facet-defining tools for the two-dimensional group problem.

Presented two-families of first known facets of two-dimensional group problem.

These new families have interesting generate stronger coefficients for continuous variable.
Thank You.