Continuous and Discontinuous Extreme Inequalities for Infinite Group Problems

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Difference Between Group Relaxation of Pure Integer and Mixed Integer Programs

1. Introduction
   - Group Relaxation of Integer Programs

2. Pure Integer Program
   - Main Results
   - Three Families of Discontinuous Extreme Inequalities

3. Mixed Integer Program
   - Continuous Extreme Inequality for MIP Group Relaxation

4. Conclusion
Infinite Group Relaxation of Integer Programs

- **Standard IP:**

\[ Ax = b \quad x \in \mathbb{Z}_+ , \]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m \times 1} \).

- Relaxation step 1: Consider each row modulo 1.

\[
\sum_{i=1}^{n} (A_{ij})(\text{mod}1)x_i \equiv b_j(\text{mod}1) \quad \forall 1 \leq j \leq m
\]

(1)

- Rewrite \( \sum_{i=1}^{n} (a_i)x_i = r \)

Each \( a_i \) belongs to the group \( \mathbb{I}^m = \{ x \in \mathbb{R}^m | 0 \leq x_i < 1 \quad \forall 1 \leq i \leq m \} \).

Note that \( a_i = (A_{i1}(\text{mod}1), \ldots, A_{im}(\text{mod}1)) \).

- Relaxation step 2: Introduce new variables.

\[
\sum_{a \in \mathbb{I}^m} ax(a) = r
\]

(2)
Infinite Group Relaxation of Integer Programs

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  \[ Ax = b \quad x \in \mathbb{Z}_+ , \]

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- **Relaxation step 1:** Consider each row modulo 1.

  \[
  \sum_{i=1}^{n} (A_{ij})(mod\,1)x_i \equiv b_j(mod\,1) \quad \forall 1 \leq j \leq m \tag{1}
  \]

- **Relaxation step 2:** Introduce new variables.

  \[
  \sum_{a \in \mathbb{I}^m} ax(a) = r \tag{2}
  \]

Each \( a_i \) belongs to the group \( \mathbb{I}^m = \{ x \in \mathbb{R}^m \mid 0 \leq x_i < 1 \quad \forall 1 \leq i \leq m \} \).

Note that \( a_i = (A_{i1}(mod\,1), \ldots, A_{im}(mod\,1)) \).
Infinite Group Relaxation of Integer Programs

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- Rewrite \( \sum_{i=1}^{n} (a_i) x_i = r \)

Each \( a_i \) belongs to the group \( \mathbb{I}^m = \{ x \in \mathbb{R}^m | 0 \leq x_i < 1 \quad \forall 1 \leq i \leq m \}. \)

Note that \( a_i = (A_{i1} \mod 1, \ldots, A_{im} \mod 1). \)

- Relaxation step 2: Introduce new variables.

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\sum_{a \in \mathbb{I}^m} ax(a) = r \tag{2}
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Finite Group Relaxation of Integer Programs

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where \( A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^{m \times 1}. \)

- Divide each row with some integer \( k \) and take modulo 1.

\[ \sum_{i=1}^{n} (A_{ij})(\text{mod}1)x_i \equiv b_j(\text{mod}1) \quad \forall 1 \leq j \leq m \quad (3) \]

- Rewrite \( \sum_{i=1}^{n} (a_i)x_i = r \)

Each \( a_i \) belongs to the group \( G = C_{|k|} \times C_{|k|} \times \ldots C_{|k|} \) where \( C_{|k|} \) is the cyclic group of order \( k. \)

- Relaxation step 2: Introduce new variables.

\[ \sum_{a \in G} ax_a = r \quad (4) \]
Finite Group Relaxation of Integer Programs

- **Standard IP:**

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Relaxation step 2: Introduce new variables.

\[ \sum_{a \in G} ax_a = r \]  

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Finite Group Relaxation of Integer Programs

- Standard IP:
  \[ Ax = b \quad x \in \mathbb{Z}_+, \]
  where \( A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^{m \times 1}. \)
- Divide each row with some integer \( k \) and take modulo 1.
  \[ \sum_{i=1}^{n} (A_{ij})(mod1)x_i \equiv b_j(mod1) \quad \forall 1 \leq j \leq m \tag{3} \]
  Rewrite \( \sum_{i=1}^{n} (a_i)x_i = r \)
  Each \( a_i \) belongs to the group \( \mathbb{G} = C_{|k|} \times C_{|k|} \ldots C_{|k|} \) where \( C_{|k|} \) is the cyclic group of order \( k \).
- Relaxation step 2: Introduce new variables.
  \[ \sum_{a \in \mathbb{G}} ax_a = r \tag{4} \]
Definition: Group Problem and Valid Inequalities

Definition (Integer Group Problem $PI(r, m)$, Johnson 1974)

For $r \in \mathbb{I}^m$ and $r \neq o$, the group problem $PI(r, m)$ is the set of functions $t : \mathbb{I}^m \rightarrow \mathbb{R}$ such that

1. $\sum_{u \in \mathbb{I}^m} ut(u) = r, r \in \mathbb{I}^m$,
2. $t(u)$ is a non-negative integer for $u \in \mathbb{I}^m$,
3. $t$ has a finite support, i.e., $t(u) > 0$ for a finite subset of $\mathbb{I}^m$.

Definition (Valid Inequality, Johnson 1974)

A function $\phi : \mathbb{I}^m \rightarrow \mathbb{R}_+$ is defined as a valid inequality for $PI(r, m)$ if $\phi(o) = 0$, $\phi(r) = 1$ and $\sum_{u \in \mathbb{I}^m} \phi(u)t(u) \geq 1, \forall t \in PI(r, m)$.

\[
\sum_{f_i \leq r} \frac{f_i}{r} x_i + \sum_{f_i > r} \frac{1 - f_i}{1 - r} x_i \geq 1
\]
Hierarchy of cutting planes

- Valid Inequality.
- **Subadditive Valid Inequality:** A function $f$ is a subadditive valid inequality if $f(x) + f(y) \geq f(x + y) \forall x, y \in \mathbb{I}^m$.
- **Minimal Inequality:** A valid inequality $\phi$ is minimal for $PI(r, m)$ if there does not exist a valid function $\phi^*$ for $PI(r, m)$ different from $\phi$ such that $\phi^*(u) \leq \phi(u) \forall u \in \mathbb{I}^m$.
- **Extreme Inequality:** A valid inequality $\phi$ is extreme for $PI(r, m)$ if whenever $\phi = \frac{1}{2} \phi_1 + \frac{1}{2} \phi_2$ for some valid inequalities $\phi_1$ and $\phi_2$ of $PI(r, m)$ then $\phi = \phi_1 = \phi_2$.

**Theorem (Gomory & Johnson 1972a,1972b, Johnson 1974)**

Valid Inequality $\supset$ Subadditive Valid Inequality $\supset$ Minimal Inequality $\supset$ Extreme Inequality
Main Contributions

All known extreme inequalities for $\mathcal{P}(r, 1)$ are continuous functions over $\mathbb{I}^1$.

Theorem (Discontinuous Extreme Inequality)

_There exists extreme inequalities for $\mathcal{P}(r, 1)$ that are discontinuous._

Other Results:

1. It is well known that piecewise linear extreme inequalities for continuous inequalities yield extreme inequalities for finite group problem. We prove a weak converse.

2. Point-wise limit of a sequences of functions that represent extreme inequalities is also extreme.
Key Result

Proposition (Continuity Proposition)

Let \( \phi : I^1 \rightarrow \mathbb{R}_+ \) be

- A piecewise linear, subadditive and valid function for \( PI(r, 1) \),
- \( \phi(u) = cu \) \( \forall u \in U \), where \( c > 0 \), \( U \equiv [0, w] \), \( w \) is a non-zero number less than 1.

Assume that

\[
\phi = (1 - \lambda)\phi_1 + \lambda\phi_2,
\]

where \( 0 < \lambda < 1 \) and \( \phi_1 \) and \( \phi_2 \) are some subadditive valid inequalities. Then \( \phi_1 \) and \( \phi_2 \) are continuous at all points at which \( \phi \) is continuous.
One Slope Extreme Inequality

The function $\pi : \mathbb{I}^1 \rightarrow \mathbb{R}_+$ is defined for a right-hand-side $r$ with $r \geq 0.5$ as

$$\pi(x) = \begin{cases} \frac{x}{r} & 0 \leq x \leq r \\ \frac{1}{2r} \left( x - \frac{x}{r} \right) & r < x < 1. \end{cases}$$

(5)

is extreme for $PI(r, 1)$.

The function $\pi$ was discovered by Letchford and Lodi(2002).

Figure: $\pi$ with $r = 0.6$
Overview of Proof: Use of Continuity Proposition

Assume by contradiction that $\pi$ is not extreme

$$\pi = \frac{1}{2} \pi_1 + \frac{1}{2} \pi_2$$

where $\pi_1 \neq \pi_2$

Using Continuity Proposition we know that $\pi_1, \pi_2$ is continuous in the intervals $[0, r]$ and $(r, 1)$.

Use Interval Lemma (Gomory and Johnson 2003):

**Proposition (Interval Lemma)**

*Let $f$ be a continuous function over intervals $U, V$ and $U + V$. If $f(u) + f(v) = f(u + v) \forall u \in U, v \in V$, $f$ is a linear function with the same slope everywhere.*

Then use this to prove that $\pi_1$ and $\pi_2$ have same slopes in the intervals $[0, r]$ and $(r, 1)$. Some algebraic manipulations lead to result. So need to know that $\pi_1$ and $\pi_2$ are continuous in the intervals $[0, r]$ and $(r, 1)$.

We obtain contradiction by proving $\pi_1 = \pi_2$. ★
Overview of Proof: Use of Continuity Proposition

- Assume by contradiction that \( \pi \) is not extreme

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\pi = \frac{1}{2} \pi_1 + \frac{1}{2} \pi_2
\]

where \( \pi_1 \neq \pi_2 \)

- Using Continuity Proposition we know that \( \pi_1, \pi_2 \) is continuous in the intervals \([0, r]\) and \((r, 1)\).

- Use Interval Lemma (Gomory and Johnson 2003):

**Proposition (Interval Lemma)**

Let \( f \) be a continuous function over intervals \( U, V \) and \( U + V \). If \( f(u) + f(v) = f(u + v) \ \forall u \in U, v \in V \), \( f \) is a linear function with the same slope everywhere.

Then use this to prove that \( \pi_1 \) and \( \pi_2 \) have same slopes in the intervals \([0, r]\) and \((r, 1)\). Some algebraic manipulations lead to result. So need to know that \( \pi_1 \) and \( \pi_2 \) are continuous in the intervals \([0, r]\) and \((r, 1)\).

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  where $\pi_1 \neq \pi_2$
- Using Continuity Proposition we know that $\pi_1, \pi_2$ is continuous in the intervals $[0, r]$ and $(r, 1)$.
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**Proposition (Interval Lemma)**

Let $f$ be a continuous function over intervals $U, V$ and $U + V$. If $f(u) + f(v) = f(u + v)$ $\forall u \in U, v \in V$, $f$ is a linear function with the same slope everywhere.

Then use this to prove that $\pi_1$ and $\pi_2$ have same slopes in the intervals $[0, r]$ and $(r, 1)$. Some algebraic manipulations lead to result. So need to know that $\pi_1$ and $\pi_2$ are continuous in the intervals $[0, r]$ and $(r, 1)$.

- We obtain contradiction by proving $\pi_1 = \pi_2$. ※
Another Interpretation: Sequence of Converging Extreme Inequalities

\[ \pi^p(x) = \begin{cases} \frac{x}{r} & x \leq r \\ 1 + \frac{2p-1}{2pr}(x-r) & r < x \leq r + p \\ \frac{x}{r} - \frac{1}{2r} & r + p < x \leq 1 - p \\ \frac{2p-1}{2pr} \left(x - 1\right) & 1 - p < x < 1 \end{cases} \]

See Dash and Günlük (2006) for an exposition of this interpretation.
Theorem (Limit Extreme Inequality)

Let \( f_i : \mathbb{I}^1 \to \mathbb{R}_+ \)

- Be piecewise linear, continuous extreme functions of \( PI(r, 1) \) for \( i \geq 1 \).
- The sequence of functions \( \{f_i\}_{i=1}^{\infty} \) converges to \( \phi \) pointwise on \( \mathbb{I}^1 \)
- \( \phi \) satisfies the conditions of Continuity Proposition
- Let \( \mathbb{G} \) be a finite subgroup of \( \mathbb{I}^1 \) such that if \( \phi \) is discontinuous at \( u \) then \( u \in \mathbb{G} \). Assume that for every \( i \in \mathbb{Z}_+ \), there is \( k(i) \in \mathbb{Z}_+ \), such that the non-differentiable points of \( f_i \) belong to \( 2^k \mathbb{G} \) and \( f_i(u) = \phi(u) \ \forall u \in 2^k \mathbb{G} \).

Then \( \phi \) is an extreme function for \( PI(r, 1) \).
Theorem

The function \( \kappa : \mathbb{I}^1 \to \mathbb{R}_+ \) is defined for \( r < 0.5 \) as

\[
\kappa(u) = \begin{cases} 
\frac{u}{r} & u \in [0, r] \\
\frac{u}{r+1} & u \in (r, 1)
\end{cases}
\]  

is extreme for \( PI(r, 1) \).

The function \( \kappa \) was discovered by Richard, Miller and Li(2006).
Two Slope Extreme Inequality - II

The function \( \zeta^\theta : I^1 \to \mathbb{R}_+ \) is defined for \( \hat{\theta} \leq \min \left\{ \frac{\hat{r}}{2}, \frac{1-\hat{r}}{4} \right\} \) as

\[
\zeta^\theta(x) = \begin{cases} 
\frac{x}{1-r-\theta} - \frac{r+\theta-x}{r} & 0 \leq x \leq r \\
\frac{1-x}{1-r} & r < x \leq r + \theta \\
\frac{\theta}{1-r} + \frac{x-1+\theta}{r} & r + \theta \leq x \leq 1 - \theta \\
1 - \theta & 1 - \theta \leq x < 1 
\end{cases}
\]  

(8)

is extreme for PI\((r, 1)\).

Figure: \( \pi \) with \( r = 0.2, \theta = 0.1 \)
What is the Merit of These Inequalities?

Definition (Gomory & Johnson 2003)

Let $C_2$ be the unit square in two dimensions. The merit index $\mathbb{M}(\phi)$ of a given inequality $\phi$ is equal to twice the area of the set of points $q \equiv (u_1, u_2) \in C_2$ such that $\phi(u_1) + \phi(u_2) = \phi(u_1 + u_2)$.

Merit Index was empirically shown to be strongly correlated to the results of the shooting experiment.

Figure: Merit Index for $\pi^p$

Similar graphs for $\kappa$ and $\zeta^\theta$. 
There exist discontinuous functions that are extreme for the integer infinite group problem. In particular we show that three different families of discontinuous valid inequalities are extreme for $PI(r, 1)$ and and the proof use different tools that we introduced.

The limiting inequality of a sequence of continuous piecewise linear extreme inequalities is extreme under some conditions.
Another Consequence of Continuity Proposition

Proposition (Finite To Infinite Group Extreme Inequality)

Let $\hat{\phi}$ be a valid subadditive extreme inequality for a finite group problem $P(C_{|G|}, r)$. Consider the linear interpolation of $\hat{\phi}$, $\phi : \mathbb{I}^1 \rightarrow \mathbb{R}_+$, defined as

$$
\phi(u) = \begin{cases} 
\hat{\phi}(u) & u \in G \\
\frac{(\hat{u}_2 - \hat{u})\hat{\phi}(u_1) + (\hat{u} - \hat{u}_1)\hat{\phi}(u_2)}{\hat{u}_2 - \hat{u}_1} & u \notin G.
\end{cases}
$$

(9)

Suppose that $\phi$ restricted to $2^kG$ is an extreme valid inequality for $P(C_{|2^kG|}, r)$ for all $k \in \mathbb{Z}_+$, then $\phi$ is extreme for the infinite group problem.

Note on Notation:

- Here $2^kG$ is a subgroup of $\mathbb{I}^1$ such that $G$ is a subgroup of $2^kG$ and

$$
\frac{|2^kG|}{|G|} = 2^k
$$

- $u_1$ and $u_2$ are the closest points of $G$ to $u$ such that $\hat{u}_1 < \hat{u} < \hat{u}_2$. 
Let $S^m$ represent the set of real $m$-dimensional vectors $w = (w_1, w_2 \ldots w_m)$, such that $\max \{||w_i||1 \leq i \leq m\} = 1$.

**Definition (Mixed Integer Group Problem MI(r,m), Johnson 1974)**

The mixed integer infinite group problem, $MI(r, m)$, is defined as a set of functions $t : \mathbb{I}^m \rightarrow \mathbb{R}$ and $s : S^m \rightarrow \mathbb{R}$ that satisfy

1. $\sum_{u \in \mathbb{I}^m} ut(u) + F(\sum_{v \in S^m} vs(v)) = r, r \in \mathbb{I}^m,$
2. $t(u)$ is a non-negative integer for $u \in \mathbb{I}^m$, $s(v)$ is a non-negative real number for $v \in S^m$,
3. $t$ and $s$ have finite supports.

**Definition (Valid Inequality)**

A valid inequality for $MI(r,m)$ is defined as a pair of functions, $\phi : \mathbb{I}^m \rightarrow \mathbb{R}_+$ and $\mu_\phi : S^m \rightarrow \mathbb{R}_+$, such that $\sum_{u \in \mathbb{I}^m} \phi(u)t(u) + \sum_{v \in S^m} \mu_\phi(v)s(v) \geq 1, \forall (t, s) \in MI(r, m)$, where $\phi(o) = 0$ and $\phi(r) = 1$. 
Difference Between Integer and Mixed Integer Infinite Group Problem

**Theorem (Johnson 1974)**

Let $\phi : \mathbb{I}^m \to \mathbb{R}_+$ and let $\tau_\phi : \mathbb{S}^m \to \mathbb{R}_+$. For any $r \in \mathbb{I} \setminus \{0\}$ and any $W \subseteq \mathbb{S}^m$, $(\phi, \tau_\phi)$ is a minimal valid inequality if and only if

\[
\begin{align*}
\phi(u) + \phi(v) & \geq \frac{\phi(u + v)}{\phi(F(hw))} \quad \forall u, v \in \mathbb{I}^m \\
\tau_\phi(w) & = \lim_{h \to 0} \frac{\phi(F(hw))}{h} \quad \forall w \in W \\
\phi(u) + \phi(u_0 - u) & = 1 \quad \forall u \in \mathbb{I}^m.
\end{align*}
\]  

(10)

**Theorem**

Let $(\phi, \mu_\phi)$ be a valid inequality for $MI(r, m)$. If $\phi$ satisfies the first two conditions in (10), then $\phi$ is continuous.

A related result was presented by Zhang (1992) assuming bounded function and a slight variant of the infinite group problem.
Final Comments

1. New method for constructing extreme inequalities for infinite group problem from extreme inequalities of finite group problems.

2. There exist discontinuous functions that are extreme for the integer infinite group problem. In particular we show that three different families of discontinuous valid inequalities are extreme for $PI(r, 1)$ and and the proof use different tools that we introduced.

3. The limiting inequality of a sequence of continuous piecewise linear extreme inequalities is extreme under some conditions.

4. We provide a proof of the fact that extreme functions for mixed integer infinite group problem are always continuous.
Thank You.