Strong mixed-integer formulations for the floor layout problem

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Abstract

The floor layout problem (FLP) tasks a designer with placing a collection of rectangular boxes on a fixed floor in such a way that minimizes total communication costs between the components. This work presents a framework for constructing mixed-integer formulations for disjunctive sets, which we apply to generate formulations and a host of valid inequalities for the FLP. We present theoretical and computational evidence for the strength of the resulting formulations.

1 Introduction

The floor layout problem (FLP), also known as the (unequal areas) facility layout problem, is central to the design of objects such as factory floors and very-large-scale integration (VLSI) computer-chips. The designer is given a fixed rectangular floor and \( N \) rectangular boxes to place onto the floor. Each box must sit completely on the floor, and they cannot overlap. Each box has a fixed area, but the widths and heights can be varied to change the shape, subject to constraints on the area and aspect ratio. The objective is to minimize the weighted sum of the Manhattan norm distances between each pair of boxes.

The FLP can be naturally formulated as a mixed-integer second-order cone program (MISOCP). However, it proven difficult to solve practical instances to optimality. In this work, we take a systematic approach to generating a family of mixed-integer programming (MIP) formulations and valid inequalities for the FLP and, more generally, disjunctive sets.

Our contributions in this work are as follows.

Formulations for the FLP

We present two new formulations for (substructures of) the pairwise FLP, from which it is straightforward to construct formulations for larger, multi-box instances. The first formulation is a strengthening of a formulation from the literature \cite{31}, and we show that it is the strongest possible formulation for a substructure of the pairwise FLP. The second formulation refines the disjunction appearing in the pairwise FLP, and reduces redundancy in the feasible set of layouts and, consequently, the branch-and-bound tree. We present computational evidence that this new “refined” formulation requires considerably fewer nodes in the branch-and-bound tree to prove optimality.
Valid inequalities for the FLP

By analyzing larger substructures of the pairwise FLP, we are able to identify a number families of valid inequalities. In particular, we present new inequalities that incorporate (a linearization of) the objective function, an important consideration as MIP formulations for the FLP have very poor relaxation bounds. We also present a procedure to lift these inequalities for the pairwise FLP to the multi-box setting and incorporate more complex positional logic between the components. We present computational evidence that these new valid inequalities strengthen the formulations considerably, and can be used to solve previously unsolved instances.

Encoding approach

To unify the formulations and inequalities presented in this work, we present a framework for constructing formulations for disjunctive sets or, equivalently, for unions of sets. The disjunctive set is embedded in a higher-dimensional space with 0/1 assigned to each branch of the disjunction to encode the different options. Our work here is a natural companion to more theoretical considerations of this approach in [42], and herein we present a simple procedure for constructing big-M formulations for disjunctive sets that we hope will be of independent interest.

2 Literature review

The floor layout problem can be viewed as a specific version of a general layout problem that consists of orthogonally packing rectangular pieces onto a rectangular floor; [21] offer a taxonomy of variations of the FLP and its relatives. Originally studied primarily in the context of factory design, the emergence of the field of very-large scale integration (VLSI) computer-chip design saw renewed interest in layout problems such as the FLP.

Broadly, algorithmic approaches to these layout problems can be grouped into two classes: exact and heuristic. Exact algorithms were predominant in the earlier literature, although the boom of applications in computer-chip design require solving large scale instances beyond the reach of existing exact approaches. As a result, a bevy of work has appeared over the past three decades, proposing heuristic approaches to produce good solutions for large-scale instances. Much of the work applies existing metaheuristic frameworks to the FLP, for example [30] and [40]. Contrastingly, many of the novel heuristics for the FLP take advantage of ideas and machinery from mathematical programming: e.g. [11, 14, 15, 24, 25, 28, 29], and [26], albeit in a way that cannot prove optimality. We note in particular the surveys of [32] and [38], which collect pointers to much of the heuristic literature.

In keeping with the MIP approach taken in this paper, we will survey the existing exact methods for the FLP in detail. Early work can be traced back to [13], who studies a discretized version of the FLP. [33] introduced a natural MIP model for the FLP, along with a collection of valid inequalities and techniques to help reduce solution time. [37] introduces novel formulations for a single pair of boxes, as well as useful computational techniques such as symmetry breaking constraints and branching priorities. [19] presents a new MIP formulation for the FLP with fewer binary variables, alongside a number of additional formulations and approaches inspired by nonlinear and mixed-integer nonlinear optimization. [31] presents another formulation inspired by a technique from [35] that reduces redundancy in the solution set. These formulations were constructed before the availability of efficient MISOCP solvers, and so careful attention has been paid on constructing and
proving desirable properties for specific linear approximations for the nonlinear area constraints in [37] and [20].

The FLP has a natural one-dimensional analogue in the single-row floor (facility) layout problem, which asks for an optimal layout of \( N \) boxes of fixed length in a straight line. This problem is already NP-hard, and strong formulations and cutting planes have been developed for the problem by [4, 2] and [5].

An intriguing line of research has investigated the FLP from the dual perspective, attempting to construct tight lower bounds. This is of particular interest for the FLP, where relaxations typically give poor bounds, even with strengthening valid inequalities. [3] presents a lower bounding technique for the single-row FLP. Another line of work investigates using semidefinite programming formulations to construct bounds for the FLP by [39] and the single-row FLP by [7]. [9] leverage the semidefinite approach to produce optimal solutions for the single-row FLP using a cutting-plane approach, and to produce high-quality solutions for larger instances in [10]. Cite our IPCO paper.

3 Preliminaries: Notation, defining constraints and basic formulations

Consider a rectangular floor \( [0, L_x] \times [0, L_y] \) for \( L_x, L_y > 0 \). There is a collection of \( N \) boxes \( \{B_i\}_{i=1}^N \) to place on the floor, each with a target area \( \alpha_i > 0 \) and maximum allowed aspect ratio \( \beta_i > 0 \). Denote the set of all pairs with \( P = \{(i, j) \in [N]^2 : i \leq j\} \) where \( [N] \triangleq \{1, \ldots, N\} \). With each pair of boxes \( (i, j) \in P \), there is an associated nonnegative unit communication cost \( p_{i,j} \). The floor layout problem then is to optimally lay out each box completely on the floor, such that the area and aspect ratio constraints are satisfied, and such that no two boxes overlap. Natural decision variables for each box \( B_i \) are the position of its center \( (c^x_i, c^y_i) \) and the lengths in each direction \( (\ell^x_i, \ell^y_i) \). The objective function used is based on the so-called “Manhattan” norm:

\[
\sum_{(i,j) \in P} p_{i,j} \left( |c^x_i - c^x_j| + |c^y_i - c^y_j| \right) .
\] (1)

Most of the constraints described above can be imposed with simple constraints. For instance, \( B_i \) lies completely on the floor iff

\[
\frac{1}{2} \ell^x_i \leq c^x_i \leq L^x - \frac{1}{2} \ell^x_i \quad \forall s \in \{x, y\}.
\] (2)

The area constraints take the form

\[
\ell^x_i \ell^y_i = \alpha_i \quad \forall i \in [N] .
\] (3)

This is a nonconvex constraint, which is typically relaxed to the constraint

\[
\ell^x_i \ell^y_i \geq \alpha_i \quad \forall i \in [N],
\] (4)

which is second-order-cone-representable [1]. The following observation states that the relaxation is valid in our setting.

Observation 1. If the costs \( p \geq 0 \), then any optimal solution for the FLP with binding area constraint (3) is also optimal for the FLP with relaxed area constraint (4).
The aspect ratio constraints take the form
\[
\min \left\{ \frac{\ell_i^x}{\ell_i^y}, \frac{\ell_i^y}{\ell_i^x} \right\} \geq \beta_i.
\] (5)

This can be represented with two linear constraints per box, but it can also be enforced on the FLP merely through bounds on the widths of the boxes.

Observation 2 (Section 2.3 of [19]). Along with the area constraints, imposing the following bounds on the box widths is sufficient to impose the aspect ratio constraints:
\[
\ell_i^s \leq \min \left\{ \sqrt{\alpha \beta_i}, \frac{L^s}{\beta_i} \right\} \overset{\text{def}}{=} ub_i^s
\] (6)
\[
\ell_i^s \geq \frac{\beta_i}{ub_i^s} \overset{\text{def}}{=} lb_i^s.
\] (7)

3.1 Disjunctive formulation for the FLP

The final constraint for the FLP requires that the boxes cannot overlap on the floor. One natural way to formulate this is by requiring each pair \( B_i \) and \( B_j \) to be separated in either the \( x \) direction or the \( y \) direction (or both).

Definition 1. We say that \( B_i \) precedes \( B_j \) in direction \( s \) if
\[
c_i^s + \frac{1}{2} \ell_i^s \leq c_j^s - \frac{1}{2} \ell_j^s.
\] (8)
We denote this with \( B_i \leftarrow_s B_j \).

Therefore, we can enforce the constraint that \( B_i \) and \( B_j \) do not overlap with the disjunctive constraint
\[
D_{i,j}^4 \overset{\text{def}}{=} \bigwedge_{k=1}^{4} d_k,
\]
where
\[
d_1 = B_i \leftarrow_y B_j
\]
\[
d_2 = B_i \leftarrow_x B_j
\]
\[
d_3 = B_j \leftarrow_y B_i
\]
\[
d_4 = B_j \leftarrow_x B_i.
\]

We omit the subscript and use \( D^4 \) when the meaning is clear from context.

Then the set of all feasible layouts is then given by the disjunctive set
\[
\mathcal{L} \overset{\text{def}}{=} \left\{ (c, \ell) \in \mathbb{R}^{2N+2N} : \begin{array}{c}
\begin{array}{c}
| 2 \ 6 \ 7 \ 4 |
\end{array}
\end{array}, \bigwedge_{i,j} D_{i,j}^4 \right\}
\] (9)

For much of the analysis to follow, we will restrict our attention to a single pair of boxes \( B_i \) and \( B_j \), in which case we will be interested in the set of feasible layouts for this pair:
\[
\mathcal{L}_{i,j} \overset{\text{def}}{=} \left\{ (c_i, c_j, \ell_i, \ell_j) \in \mathbb{R}^8 : \begin{array}{c}
\begin{array}{c}
| 2 \ 6 \ 7 \ 4 |
\end{array}
\end{array}, D^4 \right\}
\]

4
For the polyhedral analysis, we also omit the nonlinear area constraints and work instead with
\[ \hat{L}_{i,j} \overset{\text{def}}{=} \{ (c_i, c_j, \ell_i, \ell_j) \in \mathbb{R}^8 : [267], D^4 \} . \]
Equivalently, we can write this as a union of polyhedra
\[ \hat{L}_{i,j} = \bigcup_{k=1}^4 P_{i,j}^k, \]
where \( P_{i,j}^k = \{ (c_i, c_j, \ell_i, \ell_j) \in \mathbb{R}^8 : [267], d_k \} \). In the coming sections, we denote the layout problem of a single pair \( B_i \) and \( B_j \) the pairwise FLP. Much of our analysis will focus on this pairwise FLP, as in Section 6.1 we will see that it is trivial to stitch pairwise formulations together to construct formulations for our original multi-box problem.

### 3.2 Basic MIP formulations for FLP

With a disjunctive representation of the feasible region for the (pairwise) FLP, we now would like to translate the disjunctive representation to an algebraic form that off-the-shelf mixed-integer solvers accept. That is, we can construct a MIP formulation for the associated disjunctive constraint.

For simplicity, let’s first consider our options for constructing linear MIP formulations for polyhedral disjunctive representations such as \( \hat{L}_{i,j} \). Our first option is to use a classical result from [12] to construct a formulations that uses both continuous and 0/1 auxiliary variables. The resulting formulations have the favorable property that every extreme point of its LP relaxation satisfies the integrality conditions of the formulation. Such property can significantly improve the effectiveness of mixed-integer solvers and formulations that satisfy it are denoted ideal. One version of this formulation for \( \hat{L}_{i,j} \) is given in the following corollary.

**Corollary 1.** The following is an ideal formulation for \( \hat{L}_{i,j} \):

\[
\begin{align*}
\frac{1}{2} \ell^s_{p,q} & \leq c^s_{p,q} \leq L^s v_q - \frac{1}{2} \ell^s_{p,q} & \forall s \in \{x,y\}, p \in \{i,j\}, q \in [4] \quad (10a) \\
lb^s v_q & \leq \ell^s_{p,q} \leq ub^s v_q & \forall p \in \{i,j\}, q \in [4] \quad (10b) \\
c^y_{i,1} - c^y_{j,1} + \frac{1}{2} (\ell^y_{i,1} + \ell^y_{j,1}) & \leq 0 \quad (10c) \\
c^x_{i,2} - c^x_{j,2} + \frac{1}{2} (\ell^x_{i,2} + \ell^x_{j,2}) & \leq 0 \quad (10d) \\
c^y_{j,3} - c^y_{i,3} + \frac{1}{2} (\ell^y_{j,3} + \ell^y_{i,3}) & \leq 0 \quad (10e) \\
c^x_{j,4} - c^x_{i,4} + \frac{1}{2} (\ell^x_{j,4} + \ell^x_{i,4}) & \leq 0 \quad (10f) \\
\sum_{i=1}^4 c^s_{p,i} & = c^s_p & \forall s \in \{x,y\}, p \in \{i,j\} \quad (10g) \\
\sum_{i=1}^4 \ell^s_{p,i} & = \ell^s_p & \forall s \in \{x,y\}, p \in \{i,j\} \quad (10h) \\
\sum_{i=1}^4 v_i & = 1 \quad (10i) \\
v & \in \{0,1\}^4. \quad (10j)
\end{align*}
\]
Proof. This follows from Theorem 3.3 in [12].

One disadvantage of formulation (10) is the relatively large number of auxiliary variables, which can negate the computational advantage from being ideal. Additionally, this ideal property is not, in general, preserved after composition, so stitching together ideal formulations for the pairwise FLP will not give us an ideal formulation for our original multi-box problem.

A second option is to forgo idealness and construct a smaller formulation that only use 0/1 auxiliary variables (e.g. see Section 6 of [32]). For instance, using the big-M approach we obtain the following formulation for \( \hat{L}_{i,j} \).

**Lemma 1.** The following is a formulation for \( \hat{L}_{i,j} \):

\[
\begin{align*}
\frac{1}{2}\ell_k^u & \leq c_k^u \leq L^u - \frac{1}{2}\ell_k^l & \forall k \in \{i, j\} \\
\ell_k^l & \leq c_k^l \leq u\ell_k^u & \forall k \in \{i, j\} \\
\ell_k^u - \ell_k^l & + \frac{1}{2}(\ell_k^l + \ell_k^u) \leq L^u(1 - z^y_{i,j}) & \text{(11c)} \\
c_k^u - c_k^l & + \frac{1}{2}(\ell_k^l + \ell_k^u) \leq L^u(1 - z^x_{i,j}) & \text{(11d)} \\
c_k^y - c_k^l & + \frac{1}{2}(\ell_k^l + \ell_k^u) \leq L^y(1 - z^x_{j,i}) & \text{(11e)} \\
c_k^x - c_k^l & + \frac{1}{2}(\ell_k^l + \ell_k^u) \leq L^x(1 - z^y_{j,i}) & \text{(11f)} \\
z_{i,j}^x + z_{j,i}^y & + z_{i,j}^y + z_{j,i}^x = 1 & \text{(11g)} \\
z & \overset{\text{def}}{=} (z_1, z_2, z_3, z_4) & \overset{\text{def}}{=} (z^y_{i,j}, z^x_{i,j}, z^y_{j,i}, z^x_{j,i}) \in \{0, 1\}^4. & \text{(11h)}
\end{align*}
\]

Proof. First, any point satisfying (11) lies in \( \hat{L}_{i,j} \), since together, (??) imply that for any feasible solution, one component of \( z \) (w.l.o.g. \( z^y_{i,j} \)) must be equal to one, while the remaining ones are equal to zero. Then (11c) immediately implies that the nonoverlap disjunction \( D_{i,j}^4 \) is satisfied.

To see that any point in \( \hat{L}_{i,j} \) satisfies (11), note that if e.g. \( z^y_{i,j} = 0 \), then (11d) becomes \( c_k^u + \frac{1}{2}\ell_k^u \leq c_k^l - \frac{1}{2}\ell_k^x + L^x \), which is true since the stay-on-the-floor constraints (2) imply that \( c_k^x - \frac{1}{2}\ell_k^x \geq 0 \) and \( c_k^l + \frac{1}{2}\ell_k^x \leq L_x \).

Formulation (11) has fewer variables and constraints than formulation (10). However, formulation (11) is not ideal and is hence weaker than formulation (10). Fortunately, it is sometimes possible to construct ideal formulations for polyhedral disjunctive representations with sizes similar to (11) through so called embedding formulations introduced in [?].

## 4 Alternative disjunctive and MIP formulations

\( \hat{L}_{i,j} \)
4.1 Redundancy, common constraints and alternative disjunctive formulations

Consider the disjunctive constraint on variables $x \in \mathbb{R}^2$ given by

$$D^A \overset{\text{def}}{=} [x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 \leq 1] \lor [0 \leq x_2 \leq x_1 \leq 1]$$

and depicted in Figure 1(a). Because the two alternatives of $D^A$ intersect, we can check that the feasible region described by $D^A$ is identical to the feasible region described by the disjunctive constraint given by

$$D^B \overset{\text{def}}{=} [x_1 + x_2 \leq 1, \ 0 \leq x_1 \leq x_2] \lor [x_1 + x_2 \leq 1, \ 0 \leq x_2 \leq x_1] \lor [x_1 + x_2 \geq 1, \ x_2 \leq x_1 \leq 1].$$

In Section?? we will see that similar intersections between the alternatives of $\mathcal{L}_{i,j}$ lead different representations (Do we want to work with unions of polyhedra here?). Consider the disjunctive

![Figure 1: Disjunctive Constraints.](image-url)
constraint on variables \( x \in \mathbb{R}^2 \) given by

\[
D^C \overset{\text{def}}{=} \{ x \in \mathbb{R}^2 : \begin{array}{l}
x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + x_2 \leq 1 \\
0 \leq x_2 \leq x_1 \leq 2
\end{array} \} \]

and depicted in Figure 1(b) and the linear inequality

\[
x_2 \leq x_1 + \frac{1}{2}.
\] (12)

Suppose we want to construct a MIP formulation for \( M \overset{\text{def}}{=} \{ x \in \mathbb{R}^2 : (12), D^C \} \) depicted in Figure 1(c). One option is to first ignore linear inequality (12) and construct a formulation of \( \hat{M} \overset{\text{def}}{=} \{ x \in \mathbb{R}^2 : D^C \} \). For instance, as in Corollary 1, we could use the classical approach of Balas, Jeroslow and Lowe to construct the ideal formulation of \( \hat{M} \) given by

\[
\begin{align*}
x_{1,1} & \geq 0 \quad \text{(13a)} \\
x_{2,1} & \geq 0 \quad \text{(13b)} \\
x_{1,1} + x_{2,1} & \leq v_1 \quad \text{(13c)} \\
x_{2,2} & \geq 0 \quad \text{(13d)} \\
v_2 + x_{2,2} & \leq x_{1,2} \leq 2v_2 \quad \text{(13e)} \\
x_{i,1} + x_{i,2} & = x_i \quad \forall i \in [2] \quad \text{(13f)} \\
v_1 + v_2 & = 1 \quad \text{(13g)} \\
v & \in \{0,1\}^2. \quad \text{(13h)}
\end{align*}
\]

A formulation of \( M \) is then given by (13) and (12). A second option is to apply the same formulation approach to directly construct the formulation of \( M \) given by (13) and

\[
x_{2,1} \leq x_{1,1} + \frac{1}{2}v_1. \quad \text{(14)}
\]

Using basic properties of the approach of Balas, Jeroslow and Lowe we have that the projection of the LP relaxation of (13) and (14) onto \((x_1, x_2)\) is equal to the convex hull of \((M)\). In particular, any \((x_1, x_2)\) that is feasible for the LP relaxation of (13) and (14) must satisfy

\[
7x_2 \leq x_1 + 5. \quad \text{(15)}
\]

In contrast, we can check that \((x_1, x_2, x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, v_1, v_2) = (1, 1, 0, 1, 2, 1, 1, 2)\) is feasible for the LP relaxation of (13) and (12). Furthermore, given that \((x_1, x_2) = (1, 1)\) does not satisfy (15) we have that the formulation obtained by directly considering \( M \) is stronger that the one obtained by constructing a formulation for \( \hat{M} \) and then adding (12). Adding common constraints to a disjunction will always lead to equivalent or stronger formulations (assuming the same formulation approach is used), but could lead to larger and/or more complicated formulations.

A similar strengthening effect can occur when auxiliary variables used to model other aspects of a mathematical programming problem are included in the disjunctions. For instance consider the problem \( \min \{ |x - 2| : D^D \} \) for the disjunctive constraint on \( x \in \mathbb{R} \) given by \( D^D \overset{\text{def}}{=} [0 \leq x_1 \leq 1] \lor [3 \leq x_1 \leq 4] \). Using the formulation approach of Balas, Jeroslow and Lowe for \( D^D \) and a standard
LP modeling trick to linearize the absolute value in the objective we obtain the MIP formulation of this problem given by

\[
\begin{align*}
\min & \quad y_1 \\
\text{s.t.} & \quad x_1 - 2 \leq y_1 \quad (16a) \\
& \quad -x_1 + 2 \leq y_1 \quad (16b) \\
& \quad 0 \leq x_{1,1} \leq v_1 \quad (16c) \\
& \quad 3v_2 \leq x_{1,2} \leq 4v_2 \quad (16d) \\
& \quad x_{1,1} + x_{1,2} = x_1 \quad (16e) \\
& \quad v_1 + v_2 = 1 \quad (16f) \\
& \quad v \in \{0, 1\}^2. \quad (16g)
\end{align*}
\]

Alternatively, we could instead include the linearization trick in the definition of the disjunctive constraint to obtain

\[
D^E \overset{\text{def}}{=} [0 \leq x_1 \leq 1, \quad x - 2 \leq y_1, \quad -x_1 + 2 \leq y_1] \lor [3 \leq x_1 \leq 4, \quad x - 2 \leq y_1, \quad -x_1 + 2 \leq y_1],
\]

which is depicted in Figure 1(d) for the range \(y_1 \in [0, 3]\). Using the formulation approach of Balas, Jeroslow and Lowe for \(D^D\) we obtain the alternative formulation of the original problem given by

\[
\begin{align*}
\min & \quad y_1 \\
\text{s.t.} & \quad x_{1,i} - 2v_i \leq y_{1,i} \quad \forall i \in [2] \quad (17a) \\
& \quad -x_{1,i} + 2v_i \leq y_{1,i} \quad \forall i \in [2] \quad (17b) \\
& \quad y_{1,1} + y_{1,2} = y_1 \quad (16c)-(16h) \quad (17c) \\
& \quad v_1 + v_2 = 1 \quad (16g) \\
& \quad v \in \{0, 1\}^2. \quad (17d)
\end{align*}
\]

Again, using properties of the formulation approach of Balas, Jeroslow and Lowe we can check that the optimal value of the LP relaxation of (17) is equal to one. In contrast, we can also check that the optimal value of the LP relaxation of (16) is zero.

### 4.2 Selection encodings and alternative MIP formulations: the embedding approach

We now describe the procedure to construct such formulations as applicable to the FLP. In particular, we extend the procedure by introducing an adaptation of the big-M approach that yields simple and relatively strong embedding formulations, even for non-polyhedral disjunctive representations. We begin by motivating the procedure by re-interpreting (11) as a formulation for the embedding of \(\hat{L}_{i,j}\) into a higher dimensional space. Indeed, \((c_i, c_j, \ell_i, \ell_j, z)\) is feasible for (11) if and only if it belongs to the disjunctive representation given by

\[
\bigcup_{k=1}^4 P_{i,j}^k \times \{e^k\}. \quad (18)
\]
that a valid formulation for $t$ space of the Representation (18) embeds $\hat{t}_{i,j}$ for $\hat{D}_{i,j}$. Any valid formulation for this set implies a valid formulation for $\hat{L}_{i,j}$, since $\text{Proj}_{(c,\ell)} \bigcup_{k=1}^{4} \left( P_{i,j}^k \times \{ e^k \} \right) = \hat{L}_{i,j}$. However, representation [18] also makes explicit the role of the $z$ variables: $z = e^k$ only if $(c_i, c_j, \ell_i, \ell_j) \in P_{i,j}^k$. In other words, the possible values $\{ e^k \}_{k=1}^{4}$ of $z$ encode the selection among the polytopes $P_{i,j}^k$. The key for the flexibility of the embedding approach is noting that this encoding can be achieved by any family of pairwise disjoint binary vectors.

The set of feasible pairwise layouts $L_{i,j}$ can be expressed in the generic disjunctive form $\{ x \in Q : D \}$. The recipe for constructing an embedding formulation for this set takes the following ingredients:

1. A compact convex set $Q \subseteq \mathbb{R}^d$,
2. a disjunction $D = \bigvee_{k=1}^{K} [A^k x \leq b^k]$,
3. an encoding $C = \{ v^k \}_{k=1}^{K} \subseteq \{0,1\}^m$ composed of pairwise distinct 0/1-vectors (codes), and

In the hopes of constructing a formulation for the embedding

$$\text{Em}(Q, D, C) \overset{\text{def}}{=} \bigcup_{k=1}^{K} \left( Q \cap \{ x : A^k x \leq b^k \} \right) \times \{ v^k \}$$

that only uses $m + d$ variables.

If $Q$ is polyhedral, an embedding formulation from [?] is obtained if we use the unique ideal formulation of $\text{Em}(Q, D, C)$, which has $\text{Conv}(\text{Em}(Q, D, C))$ as its LP relaxation. However, the size of this formulation is strongly dependent on the encoding $C$ and could be large for all encodings. For this reason we instead adapt the big-M approach and a formulation due to [23] to obtain the following generic formulation for $\text{Em}(Q, D, C)$ and $\{ x \in Q : D \}$ that can directly deal with a non-polyhedral $Q$.

**Theorem 1.** Take $Q$ as a compact convex set, $D = \bigvee_{k=1}^{K} [A^k x \leq b^k]$, and $K$ distinct vectors $\{ v^k \}_{k=1}^{K} \subseteq \{0,1\}^m$. Take any MIP formulation $V$ for the set $C = \{ v^k \}_{k=1}^{K}$, and some affine functions $R^k_i$ such that

$$R^k_i(v^s) \begin{cases} = b^k_i & k = s \\ \geq \max_{x \in Q(s)} (A^k)x & \forall k, l, \end{cases}$$

where $Q(s) \overset{\text{def}}{=} Q \cap \{ x : A^s x \leq b^s \}$. Then

$$\begin{array}{l}
(x, v) \in Q \times V \\
(A^k)x \leq R^k_i(v) & \forall k, l
\end{array}$$  \hspace{1cm} (19)

$$\begin{array}{l}
(A^k)x \leq R^k_i(v) & \forall k, l
\end{array}$$  \hspace{1cm} (20)

is a valid formulation for $\{ x \in Q : D \}$.

**Proof.** It is clear from the definition of $D$ that for any $x \in \{ x \in Q : D \}$, there is some branch $k$ such that $A^k x \leq b^k$, and so $(x, v^k)$ is feasible for the MIP formulation by the construction of the $R^k_i$. To
show that any feasible solution for the MIP formulation lies in \( \{x \in Q : D\} \), consider some feasible \((x, v)\) for the MIP formulation. Then \( x \in Q \) and \( v \in V \) implies that \( v = v^k \) for some \( k \). Then

\[
(A^k)_{ij} x \leq b^k_j \quad \forall j,
\]

implying that \( x \) satisfies the corresponding branch \( k \) of the disjunction \( D \).

For all the encodings considered in this paper we have that \( \text{Conv} (C) \) is simple, so we always use \( V = \text{Conv} (C) \cap \mathbb{Z} \) as this yields the strongest formulation for the encoding. However, for more complex sets, constructing small formulations with similar strength may require more elaborate techniques such as those considered in [6]. Similarly, a simple set of functions

\[
R^i_j (z) = b^i_j + (M^i_j - b^i_j) \left( \sum_{k: v^i_k = 0} z_k + \sum_{k: v^i_k = 1} (1 - z^k) \right)
\]

(21)

can always be constructed for \( M^i_j \geq \max_{x \in Q} (A^i)_{ij} x \), although tighter versions are available for most of the formulations in this work.

Finally, we note that in the rest of the paper, much of the analysis will hinge on the choice of \( Q \). Since characterization of the whole pairwise FLP is intractable, especially with the inclusion of the area constraints, we instead opt to consider substructures of \( \mathcal{L}_{i,j} \), develop strong formulations, and then impose the remaining constraints and variables for our final formulation. For example, we could generate a formulation \( F \) for \( \mathcal{L}_{i,j} \), and then produce a formulation for \( \mathcal{L}_{i,j} \) as \( (c, \ell, v) \in F, [1] \).

As we will see in later sections, there is a trade-off with what to include in \( Q \), as including more constraints will lead to stronger formulations that will likely be larger and more difficult to analyze.

### 4.3 Unary formulation

We start by analyzing a simple, yet nontrivial, substructure for which we are able to construct a strong (i.e. ideal) formulation. Take

\[
Q^{lb}_{i,j} \overset{\text{def}}{=} \{(c_i, c_j, \ell_i, \ell_j) : [2, 7]\};
\]

that is, this set imposes that the boxes lie completely on the floor and lowerbounds on the box widths. For the 0/1-codes, choose the canonical unit vectors

\[
U^4 \overset{\text{def}}{=} \{e^i\}_{i=1}^4 \subseteq \{0, 1\}^4.
\]

The resulting set \( \text{Em}(Q^{lb}, D^4, U^4) \) is called the \textit{unary encoding}, as it uses one bit per branch of the disjunction.

The convex hull of the unary encoding can be represented simply, as the following theorem describes. Note that, in general, the choice of which code is assigned to which branch in the disjunction is important, and can lead to drastically different facial structure for \( \text{Conv}(\text{Em}(Q, D, U)) \). However, in the case of the unary encoding, the choices are equivalent up to permutation of the indices [7]. We include the proof for Theorem 2 in Appendix A.
Theorem 2. The following is an ideal formulation for the unary encoding $\text{Em}(Q^b, D^4, U^4)$:

\begin{align}
\frac{1}{2} \ell^x_i + lb^x_j v_2 &\leq c^x_i \leq L^x - \frac{1}{2} \ell^x_i - lb^x_j v_2 
(22a) \\
\frac{1}{2} \ell^x_j + lb^x_i v_2 &\leq c^x_j \leq L^x - \frac{1}{2} \ell^x_j - lb^x_i v_2 
(22b) \\
\frac{1}{2} \ell^y_i + lb^y_j v_3 &\leq c^y_i \leq L^y - \frac{1}{2} \ell^y_i - lb^y_j v_1 
(22c) \\
\frac{1}{2} \ell^y_j + lb^y_i v_1 &\leq c^y_j \leq L^y - \frac{1}{2} \ell^y_j - lb^y_i v_3 
(22d) \\
c^y_i + \frac{1}{2} \ell^y_i &\leq c^y_j - \frac{1}{2} \ell^y_j + L^y(1 - v_1) 
(22e) \\
c^x_i + \frac{1}{2} \ell^x_i &\leq c^x_j - \frac{1}{2} \ell^x_j + L^x(1 - v_2) 
(22f) \\
c^y_j + \frac{1}{2} \ell^y_j &\leq c^y_i - \frac{1}{2} \ell^y_i + L^y(1 - v_3) 
(22g) \\
c^x_j + \frac{1}{2} \ell^x_j &\leq c^x_i - \frac{1}{2} \ell^x_i + L^x(1 - v_4) 
(22h) \\
\ell^s_k &\geq lb^s_k \quad \forall s \in \{x, y\}, k \in \{i, j\} 
(22i) \\
\sum_{i=1}^4 v_i = 1 
(22j) \\
v \in \{0, 1\}^4. 
(22k)
\end{align}

We dub (22) the unary formulation. This formulation is similar to the FLP2 formulation used in [33] and [37], but with tightened stay-on-the-floor constraints (22a–22d).

4.4 Binary formulations

The unary encoding uses codes of length four to differentiate between four choices. If we instead use a binary encoding, we can use only two bits (i.e. codes of length two) to encode this same decision. In this setting the assignment of the codes to branches is significant (unlike the unary formulation). In particular, there are two possible encodings (up to symmetry); we focus on only one in the remainder of the section.

The gray binary encoding $\text{Em}(Q^b, D^4, GB^4)$ chooses the codes

$$GB^4 \overset{\text{def}}{=} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$ 

The name is chosen to reflect the similarities to the Gray code (see, e.g. [36]).
Proposition 1. A valid formulation for the gray binary encoding \(\text{Em}(Q^b, D^4, GB^4)\) is:

\[
\begin{align*}
\frac{1}{2} \ell^s_k & \leq c^s_k \leq L^s - \frac{1}{2} \ell^s_k & \forall s \in \{x, y\}, k \in \{i, j\} \\
\ell^s_k & \geq lb^s_k & \forall s \in \{x, y\}, k \in \{i, j\} \\
c^y_j - c^y_i & + \frac{1}{2} (\ell^y_i + \ell^y_j) \leq L^y (v_1 + v_2) \\
c^x_i - c^x_j & + \frac{1}{2} (\ell^x_i + \ell^x_j) \leq L^x (1 - v_1 + v_2) \\
c^y_i - c^y_j & + \frac{1}{2} (\ell^y_i + \ell^y_j) \leq L^y (2 - v_1 - v_2) \\
c^x_j - c^x_i & + \frac{1}{2} (\ell^x_i + \ell^x_j) \leq L^x (1 + v_1 - v_2) \\
v & \in \{0, 1\}^2
\end{align*}
\]

Proof. Apply Theorem 1 with \(D^4_{i,j}, V = \{0,1\}^2\), and \(R\) as defined in (21).

We call (23) the gray binary formulation. This gray binary formulation is reminiscent of the BLDP1 formulation from [19], although that formulation chooses the other ordering for the codes (that is, the codes for separation in each direction \(s\) differ by one bit, as opposed to the two bits in the gray binary encoding). Closer is the FLP-SP formulation in [31], although they also add the “sequence-pair” inequalities for the \(N\)-box formulation which, while strictly breaking validity, can reduce redundancy in the feasible region set without removing any valid layouts.

4.5 Refined disjunction formulation

While the disjunction \(D^4_{i,j}\) is sufficient to enforce that \(B_i\) and \(B_j\) do not overlap, its simplicity has a downside when used in a MIP framework. The disjunction is not sufficiently refined in the sense that there exist many feasible layouts that satisfy multiple branches at once. For example, in Figure 2, we see that \(B_i\) precedes \(B_j\) in both the \(x\) and \(y\) directions at once. Therefore, in any encoding constructed using \(D^4_{i,j}\), there exist two points that project down to exactly the same layout (that is, they differ only in their codes \(v\)). In practice, this redundancy can hamper the progress of branch-and-bound solvers, which must explicitly enumerate these solutions (and all nodes preceding them in the tree) to prove optimality.

To help remove this redundancy from the feasible set, we present a refined disjunction that is logically equivalent to \(D^4\). In Definition 1, we presented a linear inequality that enforces that \(B_i\) precedes \(B_j\). For our refined disjunction, we will need a description of the opposite.

Definition 2. We say that \(B_i\) does not precede \(B_j\) if

\[
c^s_i + \frac{1}{2} \ell^s_i \geq c^s_j - \frac{1}{2} \ell^s_j.
\]

We denote this with \(B_i \leftarrow \neg \neg B_j\).

Referring back to Figure 2, we see that \(B_i\) precedes \(B_k\) in direction \(y\), but does not precede \(B_k\) in direction \(x\) (and vice versa). Note in particular that, if \(c^s_i + \frac{1}{2} \ell^s_i = c^s_j - \frac{1}{2} \ell^s_j\) (as with \(B_i\) and \(B_k\) in Figure 2), we have that both \(B_i \leftarrow \neg \neg B_j\) and \(B_i \leftarrow \neg \neg B_j\) simultaneously.
Figure 2: An illustration of the possible redundancies that can be introduced when boxes can be separated in both directions. In the depiction, $B_i$ precedes $B_j$, $B_k$, and $B_t$ in direction $y$. It also precedes $B_j$ in direction $x$, does not precede $B_k$ in direction $x$, and both precedes and does not precede $B_t$ in direction $x$.

Figure 3: The eight branches of the disjunction $D^8$, illustrated via their relative position to $B_i$. 
With the two definitions, we can construct a refinement of $D_{i,j}^4$ of the form

$$D_{i,j}^8 = \bigvee_{k=1}^{8} br_k,$$

where

\begin{align*}
br_1 &= (B_i \leftarrow_y B_j) \land (B_i \leftarrow_x B_j) \land (B_j \leftarrow_x B_i) \\
br_2 &= (B_i \leftarrow_y B_j) \land (B_i \leftarrow_x B_j) \\
br_3 &= (B_i \leftarrow_x B_j) \land (B_i \leftarrow_y B_j) \land (B_j \leftarrow_y B_i) \\
br_4 &= (B_i \leftarrow_x B_j) \land (B_j \leftarrow_y B_i) \\
br_5 &= (B_j \leftarrow_y B_i) \land (B_i \leftarrow_x B_j) \land (B_j \leftarrow_x B_i) \\
br_6 &= (B_j \leftarrow_x B_i) \land (B_j \leftarrow_y B_i) \\
br_7 &= (B_j \leftarrow_x B_i) \land (B_i \leftarrow_y B_j) \land (B_j \leftarrow_y B_i) \\
br_8 &= (B_j \leftarrow_x B_i) \land (B_i \leftarrow_y B_j).
\end{align*}

We have taken a refinement of $D^4$ by splitting the regions satisfying two branches at once into the new branches $br_2, br_4, br_6,$ and $br_8$, and shrinking the other branches to exclude these new regions. See Figure 3 for an illustration.

With 8 branches in the disjunction, the encoding approach demands codes of length at least three. However, in lieu of chasing the smallest possible formulation (i.e. binary formulation), instead take the codes

\begin{align*}
C^8 \doteq \left\{ \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \right\}.
\end{align*}

Intuitively, we have taken the codes from the unary encoding for the regions $br_1, br_3, br_5,$ and $br_7$, and taken the codes for the new regions as the sum of the codes assigned to the two branches in $D^4$ the region satisfies.
Proposition 2. The following is a valid formulation for $\text{Em}(Q^{lb}, D^8, C^8)$:

\[
\begin{align*}
&\sum_{i=1}^{4} v_i \geq 1 \\
v_1 + v_3 \leq 1 \\
v_2 + v_4 \leq 1 \\
v \in \{0, 1\}^4
\end{align*}
\]

\[
c_i^y + \frac{1}{2} f_i^y + L^y v_1 \geq c_j^y - \frac{1}{2} f_j^y + (lb_i^y + lb_j^y)(v_1 + v_3)
\]

\[
c_i^x + \frac{1}{2} f_i^x + L^x v_2 \geq c_j^x - \frac{1}{2} f_j^x + (lb_i^x + lb_j^x)(v_2 + v_4)
\]

\[
c_j^y + \frac{1}{2} f_j^y + L^y v_3 \geq c_i^y - \frac{1}{2} f_i^y + (lb_i^y + lb_j^y)(v_1 + v_3)
\]

\[
c_j^x + \frac{1}{2} f_j^x + L^x v_4 \geq c_i^x - \frac{1}{2} f_i^x + (lb_i^x + lb_j^x)(v_2 + v_4).
\]

Proof. See Appendix B.

We dub this encoding $\text{Em}(Q^{lb}, D^8, C^8)$ the refined unary encoding and (25) the refined unary formulation, and conjecture that it is the strongest possible.

Conjecture 1. The formulation (25) is ideal for the refined unary encoding $\text{Em}(Q^{lb}, D^8, C^8)$.

5 Constructing valid inequalities for encodings

In Section 4 we have seen how the encoding approach can be used to construct valid formulations for substructures of the pairwise FLP. In particular, we chose a subset of variables and constraints $(Q^{lb})$ for which analysis is tractable. In this section we explore encodings of larger substructures. Since the formulations through Theorem 1 will look very similar to the ones we have seen already, we instead focus on constructing valid inequalities for these new embeddings, which in turn will give new valid inequalities for the multi-box FLP.

In the remaining sections, we will express all inequalities for the FLP in reference to the refined unary encoding. To make the notation cleaner, we will rename the 0/1 variables to

\[
\begin{pmatrix}
  z_{i,j}^y \\
z_{i,j}^x \\
z_{i,j}^y \\
z_{i,j}^x
\end{pmatrix}
\text{def}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix}
\]

with the interpretation that

\[z_{i,j}^y = 1 \implies B_i \leftarrow_s B_j.\]

This switch in notation is without loss of generality in the sense that we can construct simple affine mappings that take inequalities valid for the refined unary encoding to other embeddings. In
particular, we can take

$$A^U(v) \overset{\text{def}}{=} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$ 

and

$$A^{GB}(v) \overset{\text{def}}{=} \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} v + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$ 

and use these to map valid inequalities for the unary embedding and the gray binary embedding, respectively, to valid inequalities the refined unary embedding in the following way.

**Proposition 3.** Consider an inequality $a^T c + b^T \ell + d^T z \leq f$ with $d \geq 0$ that is valid for $\text{Em}(Q, D^8, C^8)$. Then

- $a^T c + b^T \ell + d^T A^U(v) \leq f$ is valid for $\text{Em}(Q, D^4, U^4)$, and
- $a^T c + b^T \ell + d^T A^{GB}(v) \leq f$ is valid for $\text{Em}(Q, D^4, GB^4)$.

**Proof.** We prove the first, as the second follows in exactly the same way. Consider a particular branch $br_k$ of $D^8$ and its corresponding code $v^k \in C^8$. Take now a branch $br_h$ of $D^4$ containing $br_k$ and the corresponding code $w^h \in U^4$. By construction, $A^U(w^h) \leq v^k$, and so since $d \geq 0$,

$$a^T c + b^T \ell + d^T A^U(w^h) \leq a^T c + b^T \ell + d^T v^k \leq f.$$ 

Therefore, the given inequality holds for $\text{Em}(Q, D^4, U^4)$.

This will be sufficient for the inequalities and formulations of interest here, although we note that this notion of mapping inequalities between embeddings can be generalized to other embeddings and the restriction $d \geq 0$ can be lifted; we leave this to future work.

### 5.1 Upperbound inequalities

In the previous section we chose the base set $Q^{lb}$ such that only lowerbounds on the widths were included in the formulation. This was to make the formulation analysis tractable, but enforcing the aspect-ratio constraints via (6,7) naturally includes upperbounds as well. Therefore, we can consider the set $\text{Em}(Q^{ub}, D^8, C^8)$ induced by

$$Q^{ub} \overset{\text{def}}{=} Q^{lb} \cap \{(c, \ell) \in \mathbb{R}^8 : [6]\}.$$ 

**Proposition 4.** For any assignments $\{r, s\} = \{x, y\}$ and $\{p, q\} = \{i, j\}$, then

$$c^s_p + ub^s_q(1 - z^s_{q,p}) \geq \frac{1}{2} \ell^s_p + \ell^s_q$$ 

is a valid inequality for $\text{Em}(Q^{ub}, D^8, C^8)$. If $L^s < ub^s_p + ub^s_q$,

$$z^r_{p,q} + z^r_{q,p} \geq \frac{\ell^s_p + \ell^s_q - L^s}{ub^s_p + ub^s_q - L^s}$$ 

is valid as well.
5.2 Objective inequalities

The objective (1) is nonlinear but is straightforward to linearize in the usual fashion with auxiliary variables \( \ell_{i,j}^{s}, d_{i,j}^{s} \) and the constraints

\[
\begin{align*}
\ell_{i,j}^{s} &\geq c_{i}^{s} - c_{j}^{s} \\
d_{i,j}^{s} &\geq c_{j}^{s} - c_{i}^{s}.
\end{align*}
\]

Even though this linearization is common-place, it is often not incorporated into polyhedral studies explicitly. To do this for the pairwise FLP, consider the augmented base set

\[
Q_{i,j}^{obj} = \left\{ (c_i, c_j, \ell_i, \ell_j, d_{i,j}) \in \mathbb{R}^{10} : \begin{array}{l}
\ell_{i,j}^{s} \geq 1/2 (\ell_{p}^{s} + \ell_{q}^{s}) - L^{s}(1 - z_{p,q}^{s} - z_{q,p}^{s}) \\
d_{i,j}^{s} \geq c_{p}^{s} - c_{q}^{s} + \ell_{p}^{s} + Lb_{q}^{s}(z_{p,q}^{s} + z_{q,p}^{s}) - L^{s}(1 - z_{p,q}^{s}) \\
d_{i,j}^{s} \geq c_{p}^{s} - c_{q}^{s} + Lb_{q}^{s}(z_{p,q}^{s}) \\
2d_{i,j}^{s} \geq \ell_{p}^{s} - L^{s}(1 - z_{p,q}^{s} - z_{q,p}^{s}) + Lb_{q}^{s}(z_{p,q}^{s} + z_{q,p}^{s})
\end{array} \right\}.
\]

The resulting encoding \( \text{Em}(Q_{i,j}^{obj}, D^8, C^8) \) leads to a collection of inequalities that serve to lower-bound the auxiliary objective variables \( d_{i,j}^{s} \).

**Proposition 5.** Choose \( s \in \{x, y\} \) and some assignment \( \{p,q\} = \{i,j\} \). Then the following are valid inequalities for \( \text{Em}(Q_{i,j}^{obj}, D^8, C^8) \):

\[
\begin{align*}
\ell_{i,j}^{s} &\geq 1/2 (\ell_{p}^{s} + \ell_{q}^{s}) - L^{s}(1 - z_{p,q}^{s} - z_{q,p}^{s}) \\
d_{i,j}^{s} &\geq c_{p}^{s} - c_{q}^{s} + \ell_{p}^{s} + Lb_{q}^{s}(z_{p,q}^{s} + z_{q,p}^{s}) - L^{s}(1 - z_{p,q}^{s}) \\
2d_{i,j}^{s} &\geq \ell_{p}^{s} - L^{s}(1 - z_{p,q}^{s} - z_{q,p}^{s}) + Lb_{q}^{s}(z_{p,q}^{s} + z_{q,p}^{s})
\end{align*}
\]

**Proof.** See Appendix \( \square \) D

Note that we are now adding both constraints and variables to our base-set \( Q_{i,j}^{obj} \). These inequalities are especially significant, as the MIP relaxation lower bounds for the FLP are quite poor (see Section 7.1). This is a result of the bowl-like shape of the objective, meaning the relaxation tends towards solutions that are very fractional and produce poor bounds. This phenomena continues even after branching has fixed some 0/1 variables, meaning that a solver needs to explicitly enumerate many nodes in a branch-and-bound tree before finding integer feasible solutions.

6 From pairwise to \( N \) boxes

Thus far we have only considered representations for \( \hat{L}_{i,j} \) the relationships between a single pair of boxes. In this section we address how to use the results derived for the pairwise formulations to construct strong formulations for the original \( N \)-box floor layout problem.
6.1 Multi-box formulations

It is possible to rewrite the set of feasible layouts for all \( N \) boxes purely in the disjunctive normal form

\[
\mathcal{L} \overset{\text{def}}{=} \left\{ (c, \ell) \in \mathbb{R}^{4N} : \begin{bmatrix} 2 & 6 & 7 & 4 \end{bmatrix} \right\},
\]

where \( I \) is the set of all possible configurations \((i, j; s) \in [N]^2 \times \{x, y\}\) of the \( N \) boxes that satisfy the nonoverlapping constraints. However, we leave analysis of this type to future work, and instead focus on constructing multi-box formulations in an ad-hoc way by stitching together strong pairwise formulations.

In particular, since all the constraints for the FLP involve at most two boxes, it suffices to consider each pair of boxes separately, construct a pairwise formulation, and identify all repeated variables across these pairwise formulations.

Proposition 6. Consider pairwise formulations \( F_{i,j} \) for each pair of boxes \((i, j) \in \mathcal{P}\) over the variables \((c_i, c_j, \ell_i, \ell_j, v_{i,j}) \in \mathbb{R}^8 \times \{0, 1\}^{m_{i,j}}\). If \( M \overset{\text{def}}{=} \sum_{i,j} m_{i,j} \), then the following is a formulation for the \( \mathcal{L} \):

\[
F \overset{\text{def}}{=} \left\{ (c, \ell, v) \in \mathbb{R}^{4N} \times \{0, 1\}^M : (c_i, c_j, \ell_i, \ell_j, v_{i,j}) \in F_{i,j} \quad \forall (i, j) \in \mathcal{P} \right\}.
\]

In particular, if we take the refined unary formulation for each pair of boxes, we construct the following formulation for the \( N \)-box FLP.

Corollary 2. Take \( F_{RU}^{i,j} = \{(c_i, c_j, \ell_i, \ell_j, v_{i,j}) \in \mathbb{R}^8 \times \{0, 1\}^4 : (25)\} \). Then the following is a formulation for \( \mathcal{L} \):

\[
F_{RU}^{i,j} \overset{\text{def}}{=} \left\{ (c_i, c_j, \ell_i, v_{i,j}) \in \mathbb{R}^{8N} \times \{0, 1\}^{2N(N-1)} : (c_i, c_j, \ell_i, v_{i,j}) \in F_{RU}^{i,j} \quad \forall (i, j) \in \mathcal{P} \right\}.
\]

6.2 Multi-box cutting planes

When working with more than two boxes at once, the notion of spatial transitivity appears; that is,

\[
\mathcal{B}_i \rightarrow_s \mathcal{B}_t \rightarrow_s \mathcal{B}_j \implies \mathcal{B}_i \rightarrow \mathcal{B}_j.
\]

We can use this property to generalize many of the pairwise valid inequalities introduced thus far to the multi-box setting.

Consider a pair of boxes \((i, j) \in \mathcal{P}\) and take an arbitrary path

\[
P = \{(t^0, t^1), \ldots, (t^m, t^{m+1})\} \subseteq \mathcal{P},
\]

where \( t^0 = i \) and \( t^{m+1} = j \). We define an affine function \( M^s_P \) of the following form:

\[
M^s_P(z) \overset{\text{def}}{=} \sum_{\xi=1}^{m+1} \left( z^s_{t^\xi-1, t^\xi} - 1 \right) + 1.
\]

Our function enjoys the following property:

\[
M(z) \begin{cases} = 1 & \mathcal{B}_{t^0} \leftarrow_s \mathcal{B}_{t^1} \leftarrow_s \cdots \leftarrow_s \mathcal{B}_{t^{m+1}} \\ \leq 0 & \text{otherwise} \end{cases}
\]
In essence, this will function as an (approximate) indicator for when we have a particular chain of boxes $P$ along direction $s$. We can use this to extend the logic of the pairwise inequalities we have developed. For a simple example, if $B_i \leftarrow_s B_t \leftarrow_s B_j$, then we know that $B_i$ and $B_j$ are separated in direction $s$ by at least the smallest width $B_t$ can take along that direction:

$$c_i^s + \frac{1}{2} \ell_i^s + lb_t^s \leq c_j^s - \frac{1}{2} \ell_j^s.$$

This tightening can be exploited in the inequalities derived previously, leading a host of new valid inequalities for the multi-box FLP.

**Proposition 7.** Consider the pair $(B_i, B_j)$ and an arbitrary path $P$. Choose assignments $\{r, s\} = \{x, y\}$ and $\{p, q\} = \{i, j\}$. Define

$$\gamma_P \stackrel{\text{def}}{=} \sum_{\xi=1}^{m} lb_{\xi}^s.$$

The following are valid inequalities for $F_{RU}$:

1. $$d_{i,j}^s \geq \frac{1}{2}(\ell_i^s + \ell_j^s) - L^s(1 - z_{i,j}^s - z_{j,i}^s) + \gamma_P M_p^s(z) \quad (37)$$
2. $$d_{i,j}^s \geq c_i^s - c_j^s + \ell_p^s + lb_q^s(z_{i,j}^s + z_{j,i}^s) + L^s(z_{i,j}^s - 1) + \gamma_P M_p^s(z) \quad (38)$$
3. $$d_{i,j}^s \geq c_i^s - c_j^s + (lb_i^s + lb_j^s)z_{i,j}^s + \gamma_P M_p^s(z) \quad (39)$$
4. $$2d_{i,j}^s \geq \ell_p + lb_q^s(z_{i,j}^s + z_{j,i}^s) - L^s(1 - z_{i,j}^s - z_{j,i}^s) + 2\gamma_P M_p^s(z) \quad (40)$$
5. $$c_j^s - \frac{1}{2} \ell_j^s \geq lb_i^s z_{i,j}^s + \gamma_P M_p^s(z) \quad (41)$$
6. $$c_i^s + \frac{1}{2} \ell_i^s - lb_j^s z_{i,j}^s + \gamma_P M_p^s(z) \leq L^s \quad (42)$$
7. $$c_i^s - c_j^s + \frac{1}{2}(\ell_i^s + \ell_j^s) + \gamma_P M_p^s(z) \leq L^s(1 - z_{i,j}^s). \quad (43)$$

**Proof.** See Appendix $\square$.

Proposition 7 provides an exponential number of valid inequalities for the $N$-box FLP. For small paths (e.g. $|P| = 3$), these inequalities can be added to the formulation directly. Alternatively, we provide a simple algorithm which can be used to heuristically separate them by finding a maximal path $P$ along which to separate; see Appendix $\square$.

## 7 Computational results

We compare four formulations and four levels of cuts in the computational trials. The unary formulation, denoted $U$, is based on the pairwise unary formulation (22); this is a strengthened version of the FLP2 formulation from [33]. The $BLDP1$ formulation from [20] is also tested: this is not explicitly stated in this work, but is similar to (23), with the orders of the codes $GB^4$ permuted. The third formulation is the sequence-pair formulation (SP) from [31]; this is the (23) formulation with additional constraints added derived from [35]. We note that this formulation appears to be the most performant in the literature to date. Finally, we compare with our new refined formulation (25), which we denote $RU$. 20
We will compare each of these formulations with one of four levels of valid inequalities added to the formulation (that is, they are not separated dynamically). The first will be no valid inequalities. The second will use the V2 and B2 families of inequalities appearing in [33]; the formulation name will be appended with + if these inequalities are added. The VI tag will be used for formulations with the new inequalities derived in this work added. In particular, we use (31-34), (28,29), and (41-43) for all \(|P| = 3\). Adding all of these to the formulation proved impractical in the branch-and-bound setting, so we instead add an \(O(n)\) subset of these inequalities with VI. Finally, we add (37-40) for \(|P| = 3\) and denote this by appending a 3.

For our benchmarks, we use the hp, apte, and xerox benchmarks from the MCNC benchmark collection [34]. Additionally, we will use the Armour62-1 and Armour62-2 instances from [11], the Bazaraa75-1 and Bazaraa75-2 instances from [13], the Camp91 instance from [41], and the Bozer91 from [18], and Bozer97-1, and Bozer97-2 from [17] instances. To the best of our knowledge, none of these instances have been solved to optimality before in the literature.

To construct the formulations and interface with the solver, we use the JuMP algebraic modeling language from [22, 27] in Julia (see [16]). We performed the experiments on an Intel i7-3770 3.40GHz Linux workstation with 32GB of RAM. All trials use CPLEX v12.6 with a maximum runtime of 4 hours. We also performed trials with Gurobi v6.0, but the performance was not competitive. We force CPLEX to use the linearization of the second-order cone constraints (CPX_PARAM_MIQCPSTRAT set to 2), as solving the nonlinear problem at the nodes was far slower. All trials are performed with maximum aspect ratio \(\beta = 5\).

The code used for these computational studies, as well as the benchmark instances in MPS format are available at [https://github.com/joehuchette/floor-layout](https://github.com/joehuchette/floor-layout).

### 7.1 Relaxation bound

First, we compare the lower bound produced by solving the continuous relaxation of the formulations, with and without valid inequalities added. We present the relative gap percentage \(100 \frac{U-L}{L}\), where \(L\) is the given relaxation lower bound and \(U\) is the cost of the best known feasible solution. Note in particular that a relative gap percentage of 100% implies that the relaxation bound of 0, which is the worst possible for the FLP. In Table 1 we see that all the formulations have a trivial lower bound of 0 without any inequalities added. Even after adding the symmetry breaking constraints present in [37], which partially fix the 0/1 variables for a single pair of boxes, the bound is still quite poor, and is only marginally better than the trivial bound (roughly 96% gap) on the difficult Armour62-1 and Armour62-2 instances. However, adding inequalities to the U and RU offers a substantial improvement in the relaxation gap. In particular, we can isolate the B2 inequalities from [33] and [33] as the crucial additions to the improvement in the gap. However, these inequalities do not improve the relaxation gap for the SP and BLDP1 formulations, which remain at 100% (or slightly higher, with symmetry breaking).

In Table 2, we compare the relative gap attained at the root node (with respect to the best known feasible solution). CPLEX is now able to take advantage of the added inequalities and improves the gap by roughly 1%-5% for most trials. However, there is still an appreciable difference in gap between the “four bit” formulations U and RU and the “two bit” formulations SP and BLDP1.
Table 1: Relative gap of the relaxation lower bound, with respect to the best known feasible solution. Group #1 includes U, BLDP1, BLDP1+, SP, SP+, SP+VI, SP+VI3, and RU. Group #2 includes U+, RU+VI, and RU+VI3. Symmetry breaking from [37] is added to Group #1 for comparison (# w/ SB); it does not affect the values for Group #2.

<table>
<thead>
<tr>
<th></th>
<th>#1</th>
<th>#1 w/ SB</th>
<th>#2</th>
</tr>
</thead>
<tbody>
<tr>
<td>hp</td>
<td>100%</td>
<td>89.0%</td>
<td>51.5%</td>
</tr>
<tr>
<td>apte</td>
<td>100%</td>
<td>87.5%</td>
<td>58.4%</td>
</tr>
<tr>
<td>xerox</td>
<td>100%</td>
<td>84.6%</td>
<td>56.2%</td>
</tr>
<tr>
<td>Camp91</td>
<td>100%</td>
<td>77.1%</td>
<td>43.3%</td>
</tr>
<tr>
<td>Bozer97-1</td>
<td>100%</td>
<td>88.7%</td>
<td>61.6%</td>
</tr>
<tr>
<td>Bozer97-2</td>
<td>100%</td>
<td>93.6%</td>
<td>54.6%</td>
</tr>
<tr>
<td>Bazaraa75-1</td>
<td>100%</td>
<td>85.8%</td>
<td>63.1%</td>
</tr>
<tr>
<td>Bazaraa75-2</td>
<td>100%</td>
<td>87.8%</td>
<td>68.7%</td>
</tr>
<tr>
<td>Bozer91</td>
<td>100%</td>
<td>94.4%</td>
<td>43.7%</td>
</tr>
<tr>
<td>Armour62-1</td>
<td>100%</td>
<td>96.5%</td>
<td>65.8%</td>
</tr>
<tr>
<td>Armour62-2</td>
<td>100%</td>
<td>96.8%</td>
<td>64.3%</td>
</tr>
</tbody>
</table>

Table 2: Relative gap of the root node lower bound, with respect to the best known feasible solution.

<table>
<thead>
<tr>
<th></th>
<th>U/RU</th>
<th>U+/U+VI/RU+VI</th>
<th>BLDP1</th>
<th>SP/SP+</th>
<th>SP+VI</th>
<th>SP+VI3</th>
</tr>
</thead>
<tbody>
<tr>
<td>hp</td>
<td>69.9%</td>
<td>51.5%</td>
<td>89.0%</td>
<td>88.4%</td>
<td>83.0%</td>
<td>83.7%</td>
</tr>
<tr>
<td>apte</td>
<td>72.9%</td>
<td>58.4%</td>
<td>85.7%</td>
<td>86.4%</td>
<td>83.6%</td>
<td>85.2%</td>
</tr>
<tr>
<td>xerox</td>
<td>70.3%</td>
<td>56.2%</td>
<td>84.6%</td>
<td>83.6%</td>
<td>79.1%</td>
<td>78.4%</td>
</tr>
<tr>
<td>Camp91</td>
<td>60.4%</td>
<td>40.2%</td>
<td>77.1%</td>
<td>73.3%</td>
<td>48.9%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Bozer97-1</td>
<td>74.8%</td>
<td>61.0%</td>
<td>85.5%</td>
<td>85.7%</td>
<td>79.7%</td>
<td>77.8%</td>
</tr>
<tr>
<td>Bozer97-2</td>
<td>73.2%</td>
<td>50.9%</td>
<td>93.6%</td>
<td>93.2%</td>
<td>73.9%</td>
<td>76.3%</td>
</tr>
<tr>
<td>Bazaraa75-1</td>
<td>72.6%</td>
<td>61.3%</td>
<td>85.2%</td>
<td>84.9%</td>
<td>77.9%</td>
<td>78.3%</td>
</tr>
<tr>
<td>Bazaraa75-2</td>
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<td>68.7%</td>
<td>85.0%</td>
<td>87.5%</td>
<td>82.8%</td>
<td>83.1%</td>
</tr>
<tr>
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<td>42.2%</td>
<td>94.4%</td>
<td>94.4%</td>
<td>61.7%</td>
<td>59.9%</td>
</tr>
<tr>
<td>Armour62-1</td>
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<td>65.8%</td>
<td>96.5%</td>
<td>95.3%</td>
<td>93.3%</td>
<td>91.3%</td>
</tr>
<tr>
<td>Armour62-2</td>
<td>80.5%</td>
<td>64.3%</td>
<td>96.8%</td>
<td>96.6%</td>
<td>92.0%</td>
<td>94.7%</td>
</tr>
</tbody>
</table>
7.2 Branching behavior

The rationale for introducing the refined unary formulation was that many feasible layouts will have multiple corresponding points in the encoding constructed using $D_{i,j}^4$. The refined partition removes many of these redundant solutions, which helps in the branch-and-bound setting much in the same way symmetry breaking removes equivalent feasible solutions that would otherwise have to be explicitly enumerated in the optimization procedure. Qualitatively, we observe this change of behavior in Figure 4, where we compare the progress of the SP+ and RU formulations as a function of node count on the xerox benchmark. That is, we compare both the upper and lower bound for both formulations; when they are equal, the solver has proven optimality. We see that the RU formulation requires fewer nodes to prove optimality, as expected. Additionally, we see the lower bound for the RU starts higher and improves more rapidly, as the relaxation gap results would suggest. More broadly, this illustration shows the typical trajectory when solving an instance of the FLP: finding a good (often near-optimal) feasible solution early in the procedure, and then steadily improving the lower bound which is far from optimal, until the gap is finally closed and optimality is proven.

![Figure 4](image-url)

Figure 4: Plots of lower and upper bounds on optimal cost for the sequence pair (SP+) and refined unary (RU) formulations as a function of node count.

7.3 Solution time

While the RU formulation offers advantages when looking at the progress with respect to the node count, the advantage is not so clear-cut when looking at solution time. Table 7.3 shows the solution
time for each of our benchmarks and formulation and inequality combination.

First, we see that the U and BLDP1 formulations are never competitive on the benchmark set. Some variant of the sequence pair or refined unary formulations is the best performer on all instances, with SP winning six and RU winning three. However, four of the the wins for the SP used the inequalities derived in this work. In particular we note the Bozer97-2 instance, which the sequence pair along with the new inequalities is able to solve the instance in the time limit, whereas none of the other approaches were able to close the gap beyond 3%.

The results in Table 3.3 suggests that none of the approaches herein will be a clear winner on all instances, but that some combination of them can be used to tackle difficult problems. In particular, we recommend trying both the sequence pair and refined unary formulations, along with some subsets of the inequalities derived in this work to solve the FLP instance most efficiently.

8 Conclusion and future work

In this work, we have presented a case study on systematically building strong formulations for disjunctive sets. We have used the “encoding approach” to generate mixed-integer representations for unions of sets, and constructed big-M formulations for these different embeddings. We have studied the strength of the resulting formulations, performed ad-hoc tightenings, and derived families of valid inequalities for the floor layout problem. We presented computational results where these techniques have been used to solve previously unsolved benchmark instances. Future work will also look to codify and formalize many of the techniques used in this study, as we hope they will be of general interest. Additionally, future work could investigate using more tailored approaches such as decomposition methods to solve larger-scale instances of the FLP, or at least provide lower bounds on feasible solution quality.

References


A Proof for Theorem 2

Proof. For validity, first we want to show that the stay-on-the-floor constraints can be tightened to (22a–22d) by the following case analysis. Consider (22a); the other constraints follow analogously.

- $v_2 = 0, v_4 = 0$ Reduces to the linear constraint in (2) defining $Q^{lb}$.
- $v_2 = 1, v_4 = 0$ The first inequality is unchanged. For the second, we note that $v_2 = 1$ means that

$$c_i^x + \frac{1}{2}\ell_i^x \leq c_j^y - \frac{1}{2}\ell_j^x \leq \left( L^x - \frac{1}{2}\ell_j^x \right) - \frac{1}{2}\ell_j^x \quad \text{from (2)}$$

$$= L^x - \ell_j^x \leq L^x - lb_j^x$$
• $v_2 = 0, v_4 = 1$ Same argument as the case before, but with the branch $c_j^x + \frac{1}{2}\ell_j^x \leq c_i^y - \frac{1}{2}\ell_i^x$

and the first inequality.

• $v_2 = 1, v_4 = 1$ Not feasible by (22).

To show the formulation is ideal, we want to show that every extreme point of its relaxation has integral value for $v$.

Consider the following system, which can be thought of as the projection of the relaxation of (22a-22k) onto just the $y$ variables (the argument for $x$ variables is identical). Under the assumption that $lb_i^y + lb_j^y < L^y$, the following is full-dimensional:

\[
\begin{align*}
\frac{1}{2}\ell_i + lb_j^y v_3 &\leq c_i \quad (44) \\
c_i &\leq L - \frac{1}{2}\ell_i - lb_j^y v_1 \quad (45) \\
\frac{1}{2}\ell_j + lb_i^y v_1 &\leq c_j \quad (46) \\
c_j &\leq L - \frac{1}{2}\ell_j - lb_i^y v_3 \quad (47) \\
c_i + \frac{1}{2}\ell_i &\leq c_j - \frac{1}{2}\ell_j + L^y(1 - v_1) \quad (48) \\
c_j + \frac{1}{2}\ell_j &\leq c_i - \frac{1}{2}\ell_i + L^y(1 - v_3) \quad (49) \\
\ell_i &\geq lb_i^y \quad (50) \\
\ell_j &\geq lb_j^y \quad (51) \\
v_1 &\geq 0 \quad (52) \\
v_1 &\leq 1 \quad (53) \\
v_3 &\geq 0 \quad (54) \\
v_3 &\leq 1 \quad (55) \\
v_1 + v_3 &\leq 1. \quad (56)
\end{align*}
\]

Take some arbitrary feasible $(\hat{c}, \hat{\ell}, \hat{v})$ where $0 < \hat{v}^1 < 1$ is fractional. We wish to show that this is not an extreme point. The argument for fractional $v_3$ follows in the same way. We will consider a partition of all possible cases in the following way:

1. (54) is active
   (a) (49) is active
   (b) (49) is not active

2. (54) is not active
   (a) (48) and (49) are both active
      i. (50) is active
      ii. (56) is not active
   (b) At most one of (48) and (49) are active
Note in particular that, since $\hat{v}^1 > 0$, constraint (56) immediately implies that $\hat{v}^3 < 1$ and that (52), (53) cannot be active. In each of these cases, we will argue that the solution is not extreme, either because there are not the requisite six constraints active, or because the selection leads to a contradiction.

Also, note that (45) and (46) both being active implies
\[
c_i + \frac{1}{2} \ell_i = c_j - \frac{1}{2} \ell_j + L - (lb_i + lb_j)v_1 > c_j - \frac{1}{2} \ell_j + L(1 - v_1)
\]
under the assumptions on lower bounds and on $v_1$ fractional; similarly for (44) and (47) together for $\hat{v}^3 > 0$. Therefore, these pairs cannot be active together when these conditions are met.

1.a  First consider the case that (49) and (54) are active. Clearly (55) and (56) cannot be active. Then we must have (44) and (47) active as well: take their sum, which must hold with equality as it is equivalent to (49). However, this also implies that at most three of these four active constraints are linearly independent.

We cannot have (45) or (46) active under our assumptions on the lower bounds. The only remaining possibility is if (48), (50), and (51) are all active. However, then summing (48) and (49) and reducing leads to
\[
lb_i + lb_j = L^y(2 - v_1) > L^y,
\]
a contradiction. Therefore, the point is not extreme.

1.b  Now assume that (49) is not active and (54) is. This implies that at most one of (44) and (47) can be active, as their sum is equal to (49) (when $\hat{v}^3 = 0$) as mentioned in the previous case. From our note above, at most one of (45) and (46) can be active at once.

This leaves at most three active constraints; the only possibility that the point is extreme is if all are active, along with (48), (50), and (51). If (44) and (45) are both active in this setting, their sum implies
\[
0 = L - lb_i - lb_j v_1 > L - lb_i - lb_j,
\]
a contradiction. If (44) and (46) are simultaneously active in this setting, then the sum (44) – (46) + (48) yields
\[
(L - lb_i)(1 - v_1) = 0,
\]
a contradiction. Therefore, (44) is not active. If (45) and (47) are both active in this setting, then we get a similar result for (45) – (47) – (48). This implies that we can have at most one of (44), (47) active, leaving us with too few active constraints for the point to be extreme.

2.a.i  Now consider the case that $0 < \hat{v}^3 \leq 1 - v_1 < 1$. If (48), (49), and (56) all are active, they imply
\[
\hat{\ell}_i + \hat{\ell}_j = L^y,
\]
which means that (50) and (51) cannot both be active by the assumption on lower bounds. If both (46) and (47) are active together, this implies that (51) is active; therefore, (51) is not active.
Moreover, the discussed active constraints have rank five at this point. By the notes above, (46) and (47) being active imply that (44) and (45) cannot be active. The (six) active constraints are linearly dependent, and so the point is not extreme.

The same argument holds if both (44) and (45) are active. If (44) and (46) both are active, then necessarily (45) is inactive; summing (44) and (45) yields

$$\ell_i + lb_j < L^y,$$

which says that (51) cannot be active. Presume then that (50) is active, leaving us with (44), (46), (48), (49), (50), and (56) active. Taking the resulting description for the extreme point represented by this system of active constraints yields $v_1 = \frac{-lb_i - lb_j}{L-lb_i - lb_j} < 0$, a contradiction.

The same argument holds if (45) and (47) are both active.

2.a.ii Now assume that (56) is not active and that (48) and (49) are active. Then at most one of (44) and (45) can be active, else we imply that

$$v_1 + v_3 = \frac{\ell_j}{lb_j} \geq 1,$$

which contradicts (56) not being active. The same holds for (46) and (47). Also, we cannot have both (50) and (51) active, since along with (48) and (49) active they imply

$$lb_i^y + lb_j^y = L(2 - v_1 - v_3) > L.$$

Therefore, at most five constraints are active, and we are not extreme.

2.b.i Now $0 < \delta^3 < 1$ and w.l.o.g. (49) is not active. Our statement at the beginning implies that at most two of ((44)-(47)) are active. If (56) is not active, this leaves at most five active constraints, and so the point is not extreme.

2.b.ii Now assume that (49) is not active but (56) is. By the argument in 2.b.i, there are at most six constraints ((48), (50), (51), (56), and two of (42-45)) that could be active. In particular, (48), (50), and (51) must be at an extreme point, so presume they are.

Assume for the first case that both (44) and (46) are active. Computing (44) - (46) + (48) and reducing using the other active constraints yields

$$(lb_i + lb_j - L)v_3 = 0,$$

a contradiction. Alternatively, summing (44), (47), and (48) yields

$$(lb_i + lb_j - L)(1 + v_3) = 0,$$

also a contradiction. Similarly for (45), (46), and (48), as for (45), (47), and (48). This exhausts all possible combinations of active constraints, and so the point is not extreme.
Piecing together direction-wise formulations  Now that we have established that the formulation, when restricted to a single direction, is ideal, it remains to show that the Cartesian product of formulations for both directions, along with the restriction \( v_1 + v_2 + v_3 + v_4 = 1 \), is also ideal. To see this, consider a potential fractional extreme point \((\hat{c}, \hat{\ell}, \hat{v})\) for the relaxation of the original formulation \([22a-22k]\). Then at least 12 active constraints at such an extreme point, one of which will be \([22]\). Of the 11 remaining that must exist, there will be one direction for which there are at least six active constraints. Consider three cases.

1. The direction with six active constraints has a fractional component in \(v\). Then this implies an extreme point for the auxiliary system \([44-56]\) with fractional component, a contradiction.

2. The direction with fractional component (w.l.o.g. \(y\)) has five active constraints. Then those five active constraints, along with \([56]\), induce a fractional extreme point for the auxiliary system for \(y\), a contradiction.

3. The direction with fractional component (w.l.o.g. \(y\)) has fewer than five active constraints. This implies that there are at least seven linearly independent active constraints for the auxiliary system for \(x\), a contradiction (since its dimensionality is only six).

Therefore, any fractional extreme point induces a fractional extreme point on \([44-56]\), which we have shown is impossible, and so we are done.

\[ \square \]

\[ \text{B Proof of Proposition 2} \]

Proof. The general proof technique is as follows. First, we will construct the components needed to apply Theorem 1, namely, a base-set describing shared constraints across all feasible layouts, a disjunction we are interested in modeling, a valid formulation for the codes, and some big-M functions that encapsulate the logic between the codes and the branches of the disjunction. This will leave us with a valid formulation for our set \( \mathcal{E}((\mathcal{Q}^0_d, D^8, C^8)\). We will then do ad-hoc tightening of some of the resulting constraints, giving the system described in \([25a-25b]\).

First, we choose the base-set \(\mathcal{Q}^0_d\) and the disjunction \(D^8\). We see that

\[ V \overset{\text{def}}{=} \left\{ v \in \{0, 1\}^4 : \frac{4}{i=1} v_i \geq 1, v_1 + v_3 \leq 1, v_2 + v_4 \leq 1 \right\} \]

is a valid formulation for \(C^8\). Choose big-M functions \(R\) based on the disjunction \(D^8\) in the following way. Take \(T_{p,q}\) as the \(q\)-th term of \(br_p\) (recall that \(D^8 = \bigvee_{i=1}^8 br_i\)). for example, \(T_{3,2} = B_i \leftrightarrow_y B_j\). Then take

\[ k \overset{\text{def}}{=} \begin{cases} 1 & T_{p,q} \text{ is } B_i \leftrightarrow_y B_j \text{ or } B_i \leftrightarrow_x B_j \\ 2 & T_{p,q} \text{ is } B_i \leftrightarrow_x B_j \text{ or } B_i \leftrightarrow_y B_j \\ 3 & T_{p,q} \text{ is } B_j \leftrightarrow_y B_i \text{ or } B_j \leftrightarrow_y B_i \\ 4 & T_{p,q} \text{ is } B_j \leftrightarrow_x B_i \text{ or } B_j \leftrightarrow_x B_i. \end{cases} \]

Then

\[ R_{p,q}(v) = \begin{cases} L^s(1 - v^k) & \text{the relationship in } T_{p,q} \text{ is } \leftrightarrow_s \\ (L^s - lb_i^s - lb_j^s) v^k & \text{the relationship in } T_{p,q} \text{ is } \leftrightarrow_s. \end{cases} \]
Note that, when $T_{p,q}$ is a statement of the form “$B_i$ precedes $B_j$ in direction $s$”, we get the same big-$M$ functions as appeared in the unary formulation for the same logic.

Now apply Theorem 1 and recover the valid formulation $\{(c, \ell, v) : (57 - 60), (61 - 64), (22c - 22d), (25b - 25c)\}$, where

\[
\begin{align*}
\frac{1}{2} \ell_i^x &\leq c_i^x \leq L^x - \frac{1}{2} \ell_i^x \quad (57) \\
\frac{1}{2} \ell_j^x &\leq c_j^x \leq L^x - \frac{1}{2} \ell_j^x \quad (58) \\
\frac{1}{2} \ell_i^y &\leq c_i^y \leq L^y - \frac{1}{2} \ell_i^y \quad (59) \\
\frac{1}{2} \ell_j^y &\leq c_j^y \leq L^y - \frac{1}{2} \ell_j^y \quad (60)
\end{align*}
\]

\[
\begin{align*}
c_i^y + \frac{1}{2} \ell_i^y + (L^y - lb_i^y - lb_j^y) v^1 &\geq c_j^y - \frac{1}{2} \ell_j^y \quad (61) \\
c_i^y + \frac{1}{2} \ell_i^y + (L^x - lb_i^x - lb_j^x) v^2 &\geq c_j^y - \frac{1}{2} \ell_j^y \quad (62) \\
c_j^y + \frac{1}{2} \ell_j^y + (L^y - lb_i^y - lb_j^y) v^3 &\geq c_j^y - \frac{1}{2} \ell_j^y \quad (63) \\
c_j^y + \frac{1}{2} \ell_j^y + (L^x - lb_i^x - lb_j^x) v^4 &\geq c_i^y - \frac{1}{2} \ell_i^y \quad (64)
\end{align*}
\]

Note that, since the branches of the disjunction share constraints (and corresponding big-$M$ functions $R$), many of the resulting constraints will be equivalent and duplicates can be removed.

We now wish to tighten some of these constraints by lifting them in an ad-hoc manner. By the same argument as in the proof for Theorem 2, we may tighten $(57-60)$ to $(22a-22d)$.

To tighten the new constraints $(61-64)$, we can do a case analysis. Consider $(61)$; the others follow analogously.

- $v_1 = 0, v_3 = 0$ Reduces to the linear constraint $(24)$.
- $v_1 = 1, v_3 = 0$ We have that in this case

\[
c_i^y - \frac{1}{2} \ell_i^y - (c_j^y + \ell_j^y) \geq -L^y
\]

and adding $\ell_i^y + \ell_j^y \geq lb_i^y + lb_j^y$ to this gives the desired inequality

\[
c_i^y + \frac{1}{2} \ell_i^y - (c_j^y - \frac{1}{2} \ell_j^y) \geq lb_i^y + lb_j^y - L^y.
\]

- $v_1 = 0, v_3 = 1$ In this case, we have that

\[
c_i^y - \frac{1}{2} \ell_i^y - (c_j^y + \frac{1}{2} \ell_j^y) \geq 0.
\]

We can add $\ell_i^y + \ell_j^y \geq lb_i^y + lb_j^y$ to this to get

\[
c_i^y + \frac{1}{2} \ell_i^y - (c_j^y - \frac{1}{2} \ell_j^y) \geq lb_i^y + lb_j^y
\]

the desired inequality.
- $v_1 = 1, v_3 = 1$ Not feasible by $(25c)$.
C Proof of Proposition 4

Proof. We prove by enumerating the possible values for the components of $z$ having support over the constraint, noting in particular that $z_{p,q}^s = z_{q,p}^s = 1$ is always infeasible. Recall also that

$$z_{p,q}^s = 1 \implies c_p^s + \frac{1}{2} \ell_p^s \leq c_q^s - \frac{1}{2} \ell_q^s. \quad (*)$$

(28)

- $z_{q,p}^s = 0$ Sum the constraints $c_p^s \geq \frac{1}{2} \ell_p^s$ and $ub_q^s \geq \ell_q^s$ to get the constraint $c_p^s + ub_q^s \geq \frac{1}{2} \ell_p^s + \ell_q^s$.

- $z_{q,p}^s = 1$ Using (*) and rearranging gives

$$c_p^s \geq c_q^s + \frac{1}{2} \ell_p^s + \frac{1}{2} \ell_q^s;$$

adding the constraint $c_p^s \geq \frac{1}{2} \ell_q^s$ gives the desired result

$$c_p^s \geq \frac{1}{2} \ell_p^s + \ell_q^s.$$

(29)

- $z_{p,q}^r = 0, z_{q,p}^r = 0$ Want $\ell_p^s + \ell_q^s \leq L^s$. Since $z_{p,q}^s + z_{q,p}^s + z_{p,q}^r + z_{q,p}^r \geq 1$, we must have that either $z_{p,q}^r = 1$ or $z_{q,p}^r = 1$; w.l.o.g. choose the first. Then rearranging from (*) gives

$$\frac{1}{2} (\ell_p^s + \ell_q^s) \leq c_q^s - c_p^s$$

$$\leq \left( L^s - \frac{1}{2} \ell_q^s \right) - \left( \frac{1}{2} \ell_p^s \right) \quad \text{from (2)}$$

$$\implies \ell_p^s + \ell_q^s \leq L^s.$$

- $z_{p,q}^r + z_{q,p}^r = 1$ Want that $ub_p^s + ub_q^s \geq \ell_p^s + \ell_q^s$, which follows immediately from the upper bounds on $\ell$.

\[ \square \]

D Proof of Proposition 5

Proof. We prove by enumerating the possible values for the components of $z$ having support over the constraint, noting in particular that $z_{p,q}^s = z_{q,p}^s = 1$ is always infeasible. Recall also that

$$z_{p,q}^s = 1 \implies c_p^s + \frac{1}{2} \ell_p^s \leq c_q^s - \frac{1}{2} \ell_q^s. \quad (*)$$
Want to show that $d_{i,j} \geq 1/2(\ell^s_p + \ell^s_q) - L^s$,
but it suffices to show that $\ell^s_p + \ell^s_q \leq 2L^s$, which follows immediately from summing (2) constraints $c^s_p \leq L^s - 1/2\ell^s_p$ with $1/2\ell^s_p \leq c^s_p$ to get that $\ell^s_p \leq L^s$. Applying this also for $p$ and summing the resulting inequalities gives the result.

Want to show that $d_{i,j} \geq 1/2(\ell^s_p + \ell^s_q)$,
We have from (7) that $c^s_p + 1/2\ell^s_p \leq c^s_q - 1/2\ell^s_h$; adding the appropriate constraint in (30) to this gives the result.

Same argument as the previous case.

Want to show that $d_{i,j} \geq 1/2(\ell^s_p + \ell^s_q)$,
Rearranging (8) gives $c^s_p - c^s_h + 1/2\ell^s_p + 1/2\ell^s_h \leq 0$.
Now summing the relation from (8) with one of (30) gives $\ell^s_p + \ell^s_h \leq d^s_{i,j}$.
Summing these two derived inequalities with $lb^s_h \leq \ell^s_h$ gives the desired result.

Tape one of (30) and the inequality from (8) to derive $d^s_{i,j} \geq c^s_p - c^s_h + \ell^s_p + lb^s_h - L^s$.
From the big-$M$ derived previously, we have
$L^s \geq c^s_p - c^s_h + 1/2(\ell^s_p + \ell^s_h)$;
summing the two derived inequalities along with the lower bounds on $\ell$ gives the result.
• $z_{p,q}^s = 0$ Want to show that $d_{i,j}^s \geq c_p^s - c_h^s$, which is immediate from (30).

• $z_{p,q}^s = 1$ Want to show that

$$d_{i,j}^s \geq c_p^s - c_h^s + lb_p^s + lb_h^s.$$ 

From an argument above,

$$d_{i,j}^s \geq \frac{1}{2} \ell_p^s + \frac{1}{2} \ell_h^s$$

for this particular setting, and so we are done by summing this with

$$c_p^s + \frac{1}{2} \ell_p^s \leq c_h^s - \frac{1}{2} \ell_h^s$$

implied by (1) and using the lower bounds on $\ell$.

• $z_{p,q}^s = 0, z_{q,p}^s = 0$ Want to show that

$$2d_{i,j}^s \geq \ell_p^s - L^s,$$

which just follows from the fact that $d_{i,j}^s \geq 0$ and $L^s \geq \ell_p^s$.

• $z_{p,q}^s = 1, z_{q,p}^s = 0$ Want to show that

$$d_{i,j}^s \geq \frac{1}{2} \ell_p^s + \frac{1}{2} lb_h^s,$$

which follows immediately from the inequality (valid for this particular setting for $z$)

$$d_{i,j}^s \geq \frac{1}{2} \ell_p^s + \frac{1}{2} \ell_h^s,$$

derived for a previous valid inequality.

• $z_{p,q}^s = 0, z_{q,p}^s = 1$ Same argument as the previous case.

\[\square\]

E Proof of Proposition 7

Proof. The validity argument for all the inequalities will hinge on the validity for the case $P = \{i, j\}$ (the “base case”), which has been established in the various results already discussed. We can view this technique as a lifting of the existing inequalities—developed for the pairwise formulations, but still valid for the $N$-box problem—in the direction of the other binary variables. From property (36), we immediately get validity for any binary values for $z$ except for when $z_{i,i+1}^s = 1$ for all $i \in [m+1]$, so we focus on this case. Assume that $z_{i,j}^s = 1$ as well, even though this is not necessarily implied by our condition, since this will, w.l.o.g., make the analysis simpler.
We can use the fact that \( z_{s_{i-1},i}^s = 1 \) implies \( c_{t_{i-1}}^s + \frac{1}{2} \ell_{s_{i-1}}^s \leq c_{t_i}^s - \frac{1}{2} \ell_{t_i}^s \), for all \( i \in [m+1] \) to take a telescoping sum and derive

\[
c_{t_0}^s + \frac{1}{2} \ell_{t_0}^s + \sum_{i=1}^{m} \ell_{s_i}^s \leq c_{t_{m+1}}^s - \frac{1}{2} \ell_{t_{m+1}}^s.
\]

Combining this with the lower bounds on \( \ell \) and recalling that \( t^0 = i \) and \( t^{m+1} = j \), we get

\[
c_{t_0}^s + \frac{1}{2} \ell_{t_0}^s + \gamma P \leq c_{t_{m+1}}^s - \frac{1}{2} \ell_{t_{m+1}}^s.
\]

This inequality, with \( m = 0 \), was used in the derivation of all the valid inequalities presented, so we may use this strengthening in its stead, and we are done.

\[\square\]

**F Multi-box valid inequality separation routine**

Consider a fractional solution to the FLP and pick a direction \( s \in \{x, y\} \). Construct a weighted digraph \( G \) with node-set \([N]\) and edge \((i, j)\) with weight \(-1\) if \( z_{s_{i,j}}^s = 1 \). Transitivity in direction \( s \) and the implied constraint \( z_{s_{i,j}}^s + z_{s_{j,i}}^s \leq 1 \) ensure that \( G \) is a directed acyclic graph (DAG). Weighted shortest paths on a DAG can be found in linear time \([\text{CITE}]\), and a shortest path in \( G \) will give a maximal length path \( P \). With such a \( P \), the family of cuts in Proposition 7 can be checked for violation.