**Class exercise**

Georgia Tech bookstore must decide how many 2005 calendars to order. Each calendar costs the bookstore $2 and is sold for $4.50. After January 1, any unsold calendars are returned to the publisher for a refund of 75 cents per calendar. The number of calendars sold by January 1 follows the probability distribution below. How many calendars should be ordered to maximize expected profit?

<table>
<thead>
<tr>
<th># of calendars sold</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
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<tr>
<td>Probability</td>
<td>0.30</td>
<td>0.20</td>
<td>0.30</td>
<td>0.15</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Q: number of calendars ordered  
D: demand  

\[ D \leq Q \rightarrow \text{Cost} = 2Q - 4.5D - 0.75(Q-D) = 1.25Q - 3.75D \]  
\[ D \geq Q \rightarrow \text{Cost} = 2Q - 4.5Q = -2.5Q \]

\[ c_{o} = 2 - 0.75 = $1.25 \quad cu = 4.5 - 2 = $2.5 \]

Critical ratio = \( \frac{2.5}{2.5+1.25} = \frac{2}{3} \rightarrow Q=? \)

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<td>F(Q)=P(D\leq Q)</td>
<td>0.30</td>
<td>0.50</td>
<td>0.80</td>
<td>0.95</td>
<td>1.00</td>
</tr>
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</table>
Summary - Newsvendor

- Single period
- Depending on the relationship between the cost of shortage or excess inventory, we may order more or less than expected demand
- Optimal order quantity
  - increases as shortage cost increases
  - decreases as holding cost increases
- Higher variability may cause an increase or a decrease in the optimal order quantity
  - As $\sigma$ increases, $Q^*$ will deviate more from the mean

Inventory Control - Demand

<table>
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<tr>
<th>Variability</th>
<th>Economic Order Quantity (EOQ) – Tradeoff between fixed cost and holding cost</th>
<th>Aggregate Planning – Planning for capacity levels given a forecast</th>
</tr>
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<tr>
<td>Deterministic</td>
<td></td>
<td>Materials Requirements Planning (MRP)</td>
</tr>
<tr>
<td>Stochastic</td>
<td>Lot size/Reorder level (Q,R) or (s,S) models – Tradeoff between fixed cost, holding cost, and shortage cost</td>
<td>Very difficult problem!</td>
</tr>
</tbody>
</table>

Newsvendor – single period
Lot size/Reorder level (Q,R) Models

Recap: Basic EOQ

- Place an order when the inventory level is R. The order arrives after $\tau$ time periods
  - $Q$ was the only decision variable
  - $R$ could be computed easily because $D$ was deterministic
Uncertain demand

- Both $Q$ and $R$ are decision variables
- Cycle time is no longer constant!

Inventory

(Q, R) Decisions

- We choose $R$ to meet the demand during lead time
  - Service levels: Protect against uncertainties in demand (or lead time)
  - Balance the costs: stock-outs and inventory
  - Tradeoff in $Q$: Fixed cost versus holding cost

Objective:
- Minimize
  - fixed cost + holding cost + stockout (backorder) cost
What will happen if demand follows one of these patterns?

Often the probability distribution of demand during lead time follows a Normal pattern.

\[
P(D > R) = \text{Probability of stockout}
\]

Expected demand during lead time
(Q,R) Model Assumptions

- Continuous review
- **Demand** is random and stationary. Expected demand is $d$ per unit time.
- Lead time is $\tau$
- Costs
  - $K$: Setup cost per order
  - $h$: Holding cost per unit per unit time
  - $c$: Purchase price (cost) per unit
  - $p$: Stockout cost per unit
- **Demand during lead time** is a continuous random variable $D$ with
  - pdf (density function) $f(x)$ and cdf (distribution function) $F(x)$
  - Mean=$\mu$ and standard deviation=$\sigma$

(Q,R) Model – Expected total cost per unit time

$C(Q) = h \left( s + \frac{Q}{2} \right) + \frac{K}{T} + p \frac{n(R)}{T}$

Recap: $T = \frac{Q}{d}$

$s = \text{Average inventory level before an order arrives}$

$= (\text{Reorder level}) - (\text{expected demand during leadtime}) = R - \mu$

$n(R) = \text{Expected shortage per cycle}$

$D > R \Rightarrow \text{shortage} = D - R$

$D < R \Rightarrow \text{shortage} = 0$

$n(R) = \int_0^R \int R^\infty f(x)dx + \int R^R (x - R) f(x)dx = \int_0^{\infty} (x - R) f(x)dx = \sigma L(z)$
(Q,R) Model – Expected total cost per unit time

\[ C(Q) = h \left( s + \frac{Q}{2} \right) + \frac{K}{T} + \frac{p n(R)}{T} \]

Recap: \( T = \frac{Q}{R} \)

\( s = \) Average inventory level
\( s = (\text{Reorder level}) - (\text{expected leadtime}) = R - \mu \)

\( n(R) = \) Expected shortage per cycle

\( D > R \Rightarrow \text{shortage} = D - R \)

\( D < R \Rightarrow \text{shortage} = 0 \)

\[ n(R) = \int_{0}^{R} f(x)dx + \int_{R}^{\infty} (x - R) f(x)dx \]

\[ = \int_{R}^{\infty} (x - R) f(x)dx = \frac{\sigma L(z)}{z} \]

Same expression as the “expected number of stockouts” in the newsvendor model (Q replaced by R)

\[ G(Q) = \text{Holding cost} + \text{Fixed cost} + \text{Shortage cost} \]

\[ = h \left( \frac{Q}{2} + R - d \tau \right) + K \frac{d}{Q} + p \frac{d n(R)}{Q} \]

\[ \frac{\partial G}{\partial Q} = \frac{h}{2} - \frac{Kd}{Q^2} - \frac{p d n(R)}{Q^2} = 0 \Rightarrow \frac{h}{2} = \frac{d[K + p n(R)]}{Q^2} \Rightarrow \]

\[ Q = \sqrt{\frac{2d[K + p n(R)]}{h}} \]

\[ \frac{\partial G}{\partial R} = h - \frac{p d (1 - F(R))}{Q} \Rightarrow 1 - F(R) = \frac{Qh}{pd} \]
(Q,R) Model – Expected total cost per unit time

\[ C(Q) = \text{Holdingcost} + \text{Fixedcost} + \text{Shortagecost} \]

\[ = h\left(\frac{Q}{2} + R - d \tau\right) + K \frac{d}{Q} + p \frac{d \, n(R)}{Q} \]

Optimalsolution:

1. \[ Q = \sqrt{\frac{2d[K + p \, n(R)]}{h}} \]
2. \[ F(R) = 1 - \frac{Qh}{pd} \]

How do we pull Q and R from these equations? \(\Rightarrow\) Solve iteratively!!

Solving for optimal Q and R

- Start with a \(Q_0\) value and iterate until the Q values converge

Remember: To find Q, you need \(n(R) = \sigma L(z)\)
Lookup for z in the Normal tables
Example – Rainbow Colors

Rainbow Colors paint store uses a \((Q,R)\) inventory system to control its stock levels. For a popular eggshell latex paint, historical data show that the distribution of monthly demand is approximately Normal, with mean 28 and standard deviation 8. Replenishment lead time for this paint is about 14 weeks. Each can of paint costs the store $6. Although excess demands are backordered, each unit of stockout costs about $10 due to bookkeeping and loss-of-goodwill. Fixed cost of replenishment is $15 per order and holding costs are based on a 30% annual interest rate.

- What is the optimal lot size (order quantity) and reorder level?
- What is the expected inventory level (safety stock) just before an order arrives?

Example – Rainbow Colors

Input
- Monthly demand Normal mean=28 std.dev.=8
- \(\tau\) = 14 weeks
- \(c=6\), \(p=10\), \(K=15\)
- \(h=ic=(0.3)(6)=1.8/\text{unit/year}\)

Computed input
- \(d=\) ? (Expected annual demand)
- Expected demand during lead time

\[ \mu = ? \]

- Variance of demand during lead time

\[ \sigma^2 = ? \]
Example – Rainbow Colors

- **Input**
  - Monthly demand Normal  mean=28  std.dev.=8
  - τ=14 weeks
  - c=$6,  p=$10,  K=$15
  - h=ic=(0.3)(6)=$1.8/unit/year

- **Computed input**
  - d=(28)(12)=336 units/year  (Expected annual demand)
  - Expected demand during lead time
    \[
    \mu = \frac{336 \text{ units/year} \times (14 \text{ weeks})}{52 \text{ weeks/year}} = 90 \text{ units}
    \]
  - Variance of demand during lead time
    \[
    \text{Annual variance} = (12)(8^2) = 768
    \]
    \[
    \text{Variance of lead time demand} = 768 \times \frac{14}{52} = 206.77 \Rightarrow \sigma = 14.38
    \]

As the lead time increases, so does the mean and variance of demand during lead time

Shorter lead times \(\iff\) Less variability of demand during lead time

\[
\mu = \frac{(28)(12) \text{ units/year} \times (14 \text{ weeks})}{52 \text{ weeks/year}} = 90 \text{ units}
\]

\[
\text{Variance of demand during lead time}
\]

\[
\text{Annual variance} = (12)(8^2) = 768
\]

\[
\text{Variance of lead time demand} = 768 \times \frac{14}{52} = 206.77 \Rightarrow \sigma = 14.38
\]
Example – Rainbow Colors

- **Iteration 0**: Compute EOQ

  \[
  Q_0 = \sqrt{\frac{2Kd}{h}} = \sqrt{\frac{(2)(15)(336)}{1.8}} = 75
  \]

- **Iteration 1**: Compute \( R_1 \) (given \( Q_0 \)) and then compute \( Q_1 \) (given \( R_1 \))

  \[
  F(R_1) = 1 - \frac{Q_0 h}{pd} = 1 - \frac{(75)(1.8)}{(10)(336)} = 0.96 = \Phi(z) \Rightarrow z = 1.75
  \]

  From standard Normal table

  \[
  z = \frac{R - \mu}{\sigma} \Rightarrow R = \sigma z + \mu = R_1 = (14.38)(1.75) + 90 \approx 115
  \]

  Safety Stock

  Expected Demand during Lead time
Example – Rainbow Colors

- **Iteration 1 (continued):** Compute $Q_1$ (given $R_1$)
  
  $Q = \sqrt{\frac{2d[K + p n(R)]}{h}}$
  
  $n(R_1) = \sigma L(z) = (14.38)(0.0162) = 0.233$
  
  $Q_1 = \sqrt{\frac{(2)(336)[15 + (10)(0.233)]}{1.8}} \approx 80$

  $Q_0 = 75$
  
  $Q_1 = 80$
  
  $R_1 = 115$

  $Q_0$ and $Q_1$ not close, continue iterations

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Example – Rainbow Colors

- **Iteration 2:** Compute $R_2$ (given $Q_1$) and then compute $Q_2$ (given $R_2$)
  
  $F(R_2) = 1 - \frac{Q_1 h}{pd} = 1 - \frac{(80)(1.8)}{(10)(336)} = 0.957 = \Phi(z) \Rightarrow z = 1.72$
  
  $R = \sigma z + \mu = R_2 = (14.38)(1.72) + 90 = 115$

  $Q_0 = 75$
  
  $Q_1 = 80$
  
  $R_1 = 115$
  
  $R_2 = 115$

  STOP! R values converged, optimal $(Q,R) = (80,115)$