Solutions to Homework 2

1) We have the optimality equations as
\[ V^*(s_1) = \max \{ 1 + 2V^*(s_1), 2V^*(s_2) \} \]

\[ V^*(s_2) = 2 + 2V^*(s_2) \implies V^*(s_2) = \frac{2}{1-\gamma} \]

Thus, \( \alpha_1 \) to be the optimal action in state \( s_1 \), we need

\[ 1 + 2V^*(s_1) \geq \frac{2\gamma}{1-\gamma} \quad \text{in which case} \quad V^*(s_1) = \frac{1}{1-\gamma} \]

Therefore, we need

\[ 1 + \frac{2\gamma}{1-\gamma} \geq \frac{2\gamma}{1-\gamma} \implies \gamma < 0.5 \]

For \( \alpha_2 \) to be optimal in state \( s_2 \), we need

\[ 1 + 2V^*(s_1) \leq \frac{2\gamma}{1-\gamma} \quad \text{in which case} \quad V^*(s_1) = \frac{2\gamma}{1-\gamma} \]

Thus, we need

\[ 1 + \frac{2\gamma^2}{1-\gamma} \leq \frac{2\gamma}{1-\gamma} \text{ which implies that } \gamma \geq 0.5 \]

2) Let \( f \) and \( g \) be two superadditive functions. Then

\[ f(x^+, y^+) + f(x^-, y^-) > f(x^+, y^-) + f(x^-, y^+) \quad \text{and} \]

\[ g(x^+, y^+) + g(x^-, y^-) > g(x^+, y^-) + g(x^-, y^+) \]

It is clear from the above inequalities that

\[ f(x^+, y^+) + g(x^+, y^+) + f(x^-, y^-) + g(x^-, y^-) > f(x^+, y^-) + g(x^+, y^-) + f(x^-, y^+) + g(x^-, y^+) \]

which completes the proof.
3. Let \( x^+ \geq x^- \) and select \( y \geq f(x^+) \). From the definition of \( f \), we have

\[
g(x^+, f(x^+)) - g(x^+, f(x^-)) \geq 0.
\]

Since \( g \) is subadditive,

\[
g(x^+, f(x^+)) + g(x^-, f(x^-)) \leq g(x^+, f(x^+)) + g(x^+, f(x^-)) \quad \text{which implies that}
\]

\[
g(x^+, f(x^-)) - g(x^+, f(x^-)) \leq g(x^+, f(x^-)) - g(x^+, f(x^-)).
\]

Thus, \( g(x^+, f(x^-)) - g(x^+, f(x^-)) \geq 0 \) for all \( y \geq f(x^-) \).

But since \( g(x^+, f(x^+)) \geq g(x^+, f(x^-)) \) we must have \( g(x^+) \leq f(x) \).

4. We will first do an induction on \( S \) and then an induction on \( A \).

Since \( g(s+1, a+1) + g(s, a) \geq g(s+1, a) + g(s, a+1) \), we have

\[
g(s+2, a+2) + g(s+1, a) \geq g(s+2, a+1) + g(s+1, a+1)
\]

for \( n = 1 \). Now assume that

\[
g(s+n-1, a+1) + g(s, a) \geq g(s+n, a+1) + g(s+n-1, a).
\]  \( (1) \)

But we also know that

\[
g(s+n, a+1) + g(s+n-1, a) \geq g(s+n, a+1) + g(s+n, a).
\]  \( (2) \)

Summing up \( (1) \) and \( (2) \) yields

\[
g(s+n, a+1) + g(s, a) \geq g(s+n, a+1) + g(s+n, a) \quad \text{for} \quad n \geq 1.
\]  \( (3) \)

Now assume that \( g(s+n, a+m+1) + g(s, a) \geq g(s, a+m+1) + g(s+n, a) \). We know that

\[
g(s+n, a+m) + g(s, a+m+1) \geq g(s, a+m) + g(s+n, a+m-1)
\]

Adding these two inequalities, we have

\[
g(s+n, a+m) + g(s, a) \geq g(s, a+m) + g(s+n, a) \quad \text{and} \quad n \geq 1 \text{ and } m \geq 1.
\]