Name and student number: __________________

Grade:________

1. [25] Sam, John and Bill enter a barbershop simultaneously and served immediately, Sam to get a shave, John a haircut and Bill both. The amount of time it takes to receive the service is exponentially distributed with means of 15 minutes for shave, 20 minutes for haircut and 30 minutes for both. Find the probability that John finishes first.

Solution: Let \( S \), \( H \), and \( B \) denote the times for a shave, a haircut, and both, respectively. Their respective rates are \( \lambda_1 = 4 \) per hour, \( \lambda_2 = 3 \) per hour, and \( \lambda_3 = 2 \) per hour. Then we want to find

\[
P\{H \leq \min(S, B)\} = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{3}{4 + 3 + 2} = \frac{1}{3}.
\]

Note that \( \min(S, B) \sim \text{Exp}(\lambda_1 + \lambda_3) \).

2. [35] Customers arrive to a bank at a Poisson rate of 3 per hour. Suppose two customers arrived during the first hour. What is the probability that: (i) both arrived during the first 30 minutes? (ii) at least one arrived during the first 30 minutes?

Solution: (i) Given a certain number of events from a Poisson process occurred in a given time interval, the times of these events are uniformly distributed. For our problem, we have two iid uniform rv’s. The probability that both took place in the first half of the interval is \( 1/4 \).

(ii) By the same reasoning as in (i) above, the probability that at least one arrived within the first thirty minutes is \( 1 - P\{\text{both arrived in the last thirty minutes}\} = 3/4 \).

3. [40] A service center consists of two servers, each working at an exponential rate of three services per hour. Customers arrive at a Poisson rate of 8 per hour, and the system capacity is at most 4 customers. Find: (i) Fraction of potential customers which enter the system. (ii) The average number of customers in the system. (iii) The average revenue (per hour) of the center assuming that each served customer pays 10\$ for the service.

Solution: (i) Note this is an \( M/M/2/4 \) queue with arrival rate \( \lambda = 8 \) per hour and service rates of 3 per hour. Solving the balance equations for our stationary distribution \( p = [p_0, p_1, p_2, p_3, p_4] \) we get

\[
p_0 = \frac{81}{1481} = 0.055, \quad p_1 = \frac{216}{1481} = 0.146, \quad p_2 = \frac{288}{1481} = 0.194, \quad p_3 = \frac{384}{1481} = 0.259, \quad p_4 = \frac{512}{1481} = 0.346.
\]
The fraction of arrivals that enter the system is \( 1 - p_4 = 0.654 \), since only when at capacity customers are turned away. 

(ii) The average number of customers in the system in steady state is 
\[
\sum_{i=0}^{4} ip_i = 2.695. 
\]

(iii) The rate at which customers enter (and hence leave) the system in the steady state is 
\[
\lambda^* = (1 - p_4)\lambda = 5.23. 
\] All customers that actually enter the system pay $10 for service. Since the queue is stable, the average revenue rate is $10\lambda^* = $52.3 per hour.