Question 1

Below is a network, with costs and travel times indicated on the arcs, in the format (cost, travel time), and load sizes, service times, earliest service start time, and latest service completion time indicated next to the nodes in the format (load size, service time), [earliest service start time, latest service completion time]. Node 0 is the depot. You have to find the optimal set of vehicle routes that start at the depot and return to the depot, and that serve all the customers within the time windows specified. Each vehicle has a capacity of 100 units. A customer must be served by exactly one vehicle (split deliveries are not allowed). You have to use the minimum number of vehicles.

1. Write down a complete integer linear programming formulation for the vehicle routing problem on the network. With “complete” is meant that you should write the objective function and each individual constraint with the given data substituted in the appropriate places.

**Answer:** Let $N = \{0, 1, 2, 3, 4, 5, 6\}$ denote the set of nodes. Let $A = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}, \{0, 6\}, \{1, 0\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}, \{2, 0\}, \{2, 1\}, \{2, 3\}, \{2, 4\}, \{2, 6\}, \{3, 0\}, \{3, 1\}, \{3, 2\}, \{3, 4\}, \{3, 5\}, \{4, 0\}, \{4, 2\}, \{4, 3\}, \{4, 5\}, \{4, 6\}, \{5, 0\}, \{5, 1\}, \{5, 3\}, \{5, 4\}, \{5, 6\}, \{6, 0\}, \{6, 1\}, \{6, 2\}, \{6, 4\}, \{6, 5\}\}$ denote the set of arcs.

Let $G = (N, A)$ denote the given directed network where node 0 is the depot. Let $c_{ij}$ and $t_{ij}$ denote the cost and the travel time associated with arc $\{i, j\}$. Let $d_i, s_i, a_i$, and $b_i$ denote respectively the load size, service time, earliest service start time, and latest service completion time for customer $i$.

For any subset $S$ of nodes, $S \subset N$, let $\alpha(S)$ denote the minimum number of vehicles needed to serve all customers in $S$.

**Decision variables:**

Let $x_{ij} = \begin{cases} 1 & \text{if a vehicle travels on arc } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$

Let $y_i = \text{Service start time at customer } j$.

Then the formulation of the problem is as follows:

\[
\begin{align*}
\min & \quad \sum_{\{i, j\} \in A} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{\{j \in N: \{i, j\} \in A\}} x_{ij} = 1 & \forall i \in N \setminus \{0\} \\
& \quad \sum_{\{j \in N: \{j, i\} \in A\}} x_{ji} = 1 & \forall i \in N \setminus \{0\} \\
& \quad \sum_{\{(i, j) \in A: i \in S, j \not\in S\}} x_{ij} \geq \alpha(S) & \forall S \subset N \setminus \{0\} : 2 \leq |S| \leq |N \setminus \{0\}| \\
& \quad y_i \geq a_i \\& \quad y_i + s_i \leq b_i \\& \quad t_{0j} \leq y_j + M (1 - x_{0j}) & \forall j \in N : \{0, j\} \in A \\
& \quad y_i + s_i + t_{ij} \leq y_j + M (1 - x_{ij}) & \forall \{i, j\} \in A : i \neq 0, j \neq 0 \\
& \quad x_{ij} \in \{0, 1\} & \forall \{i, j\} \in A
\end{align*}
\]
With the given data, the formulation is

\[
\begin{align*}
\min \quad & \{12x_{01} + 26x_{02} + 18x_{03} + 18x_{04} + 17x_{05} + 21x_{06} \\
& + 11x_{10} + 27x_{12} + 28x_{13} + 25x_{15} + 33x_{16} + 26x_{20} \\
& + 26x_{21} + 31x_{23} + 18x_{24} + 19x_{26} + 17x_{30} + 29x_{31} \\
& + 30x_{32} + 21x_{34} + 16x_{35} + 19x_{40} + 19x_{42} + 22x_{43} \\
& + 31x_{45} + 17x_{46} + 18x_{50} + 24x_{51} + 17x_{53} + 32x_{54} \\
& + 34x_{56} + 22x_{60} + 32x_{61} + 18x_{62} + 18x_{64} + 34x_{65}\} \\
\text{s.t.} \quad & x_{10} + x_{12} + x_{13} + x_{15} + x_{16} = 1 \\
& x_{01} + x_{21} + x_{31} + x_{51} + x_{61} = 1 \\
& x_{20} + x_{21} + x_{23} + x_{24} + x_{26} = 1 \\
& x_{02} + x_{12} + x_{32} + x_{42} + x_{62} = 1 \\
& x_{30} + x_{31} + x_{32} + x_{34} + x_{35} = 1 \\
& x_{03} + x_{13} + x_{23} + x_{43} + x_{53} = 1 \\
& x_{40} + x_{42} + x_{43} + x_{45} + x_{46} = 1 \\
& x_{04} + x_{24} + x_{34} + x_{54} + x_{64} = 1 \\
& x_{50} + x_{51} + x_{53} + x_{54} + x_{56} = 1 \\
& x_{05} + x_{15} + x_{35} + x_{45} + x_{65} = 1 \\
& x_{60} + x_{61} + x_{62} + x_{64} + x_{65} = 1 \\
& x_{06} + x_{16} + x_{26} + x_{46} + x_{56} = 1 \\
& x_{10} + x_{13} + x_{15} + x_{16} + x_{20} + x_{23} + x_{24} + x_{26} \geq 1 \quad S = \{1, 2\} \\
& x_{10} + x_{12} + x_{15} + x_{16} + x_{30} + x_{32} + x_{34} + x_{35} \geq 1 \quad S = \{1, 3\} \\
& x_{10} + x_{12} + x_{13} + x_{15} + x_{16} + x_{40} + x_{42} + x_{43} + x_{45} + x_{46} \geq 1 \quad S = \{1, 4\} \\
& x_{10} + x_{12} + x_{13} + x_{16} + x_{50} + x_{53} + x_{54} + x_{56} \geq 1 \quad S = \{1, 5\} \\
& x_{10} + x_{12} + x_{13} + x_{15} + x_{60} + x_{62} + x_{64} + x_{65} \geq 1 \quad S = \{1, 6\} \\
& x_{20} + x_{21} + x_{24} + x_{26} + x_{30} + x_{31} + x_{34} + x_{35} \geq 1 \quad S = \{2, 3\} \\
& x_{20} + x_{21} + x_{23} + x_{24} + x_{26} + x_{40} + x_{43} + x_{45} + x_{46} \geq 1 \quad S = \{2, 4\} \\
& x_{20} + x_{21} + x_{23} + x_{24} + x_{26} + x_{50} + x_{51} + x_{53} + x_{54} + x_{56} \geq 1 \quad S = \{2, 5\} \\
& x_{20} + x_{21} + x_{23} + x_{24} + x_{26} + x_{60} + x_{61} + x_{64} + x_{65} \geq 1 \quad S = \{2, 6\} \\
& x_{30} + x_{31} + x_{32} + x_{35} + x_{40} + x_{42} + x_{45} + x_{46} \geq 1 \quad S = \{3, 4\} \\
& x_{30} + x_{31} + x_{32} + x_{34} + x_{50} + x_{51} + x_{54} + x_{56} \geq 1 \quad S = \{3, 5\} \\
& x_{30} + x_{31} + x_{32} + x_{34} + x_{35} + x_{60} + x_{61} + x_{62} + x_{64} + x_{65} \geq 1 \quad S = \{3, 6\} \\
& x_{40} + x_{42} + x_{43} + x_{46} + x_{50} + x_{51} + x_{53} + x_{56} \geq 1 \quad S = \{4, 5\} \\
& x_{40} + x_{42} + x_{43} + x_{45} + x_{60} + x_{61} + x_{62} + x_{65} \geq 1 \quad S = \{4, 6\} \\
& x_{50} + x_{51} + x_{53} + x_{54} + x_{60} + x_{61} + x_{62} + x_{64} \geq 1 \quad S = \{5, 6\} \\
& x_{10} + x_{15} + x_{16} + x_{20} + x_{24} + x_{26} + x_{30} + x_{34} + x_{35} \geq 1 \quad S = \{1, 2, 3\} \\
& x_{10} + x_{13} + x_{15} + x_{16} + x_{20} + x_{23} + x_{26} + x_{40} + x_{43} + x_{45} + x_{46} \geq 1 \quad S = \{1, 2, 4\} \\
& x_{10} + x_{13} + x_{16} + x_{20} + x_{23} + x_{24} + x_{26} + x_{50} + x_{53} + x_{54} + x_{56} \geq 1 \quad S = \{1, 2, 5\} \\
& x_{10} + x_{13} + x_{15} + x_{20} + x_{23} + x_{24} + x_{60} + x_{64} + x_{65} \geq 1 \quad S = \{1, 2, 6\} \\
& x_{10} + x_{12} + x_{15} + x_{16} + x_{30} + x_{32} + x_{35} + x_{40} + x_{42} + x_{45} + x_{46} \geq 1 \quad S = \{1, 3, 4\} \\
& x_{10} + x_{12} + x_{16} + x_{30} + x_{32} + x_{34} + x_{50} + x_{54} + x_{56} \geq 1 \quad S = \{1, 3, 5\} \\
& x_{10} + x_{12} + x_{15} + x_{30} + x_{32} + x_{34} + x_{35} + x_{60} + x_{62} + x_{64} + x_{65} \geq 1 \quad S = \{1, 3, 6\}
\end{align*}
\]
\[\begin{align*}
x_{10} + x_{12} + x_{13} + x_{16} + x_{40} + x_{42} + x_{43} + x_{46} + x_{50} + x_{53} + x_{56} & \geq 1 \quad S = \{1, 4, 5\} \\
x_{10} + x_{12} + x_{13} + x_{15} + x_{40} + x_{42} + x_{43} + x_{45} + x_{60} + x_{62} + x_{65} & \geq 1 \quad S = \{1, 4, 6\} \\
x_{10} + x_{12} + x_{13} + x_{50} + x_{53} + x_{54} + x_{60} + x_{62} + x_{64} & \geq 1 \quad S = \{1, 5, 6\} \\
x_{20} + x_{21} + x_{26} + x_{30} + x_{31} + x_{35} + x_{40} + x_{45} + x_{46} & \geq 1 \quad S = \{2, 3, 4\} \\
x_{20} + x_{21} + x_{24} + x_{26} + x_{30} + x_{31} + x_{34} + x_{50} + x_{51} + x_{54} + x_{56} & \geq 1 \quad S = \{2, 3, 5\} \\
x_{20} + x_{21} + x_{24} + x_{30} + x_{31} + x_{34} + x_{35} + x_{60} + x_{61} + x_{64} & \geq 1 \quad S = \{2, 3, 6\} \\
x_{20} + x_{21} + x_{23} + x_{26} + x_{40} + x_{43} + x_{46} + x_{50} + x_{51} + x_{53} + x_{56} & \geq 1 \quad S = \{2, 4, 5\} \\
x_{20} + x_{21} + x_{23} + x_{40} + x_{43} + x_{45} + x_{60} + x_{61} + x_{65} & \geq 1 \quad S = \{2, 4, 6\} \\
x_{20} + x_{21} + x_{23} + x_{50} + x_{51} + x_{53} + x_{54} + x_{60} + x_{61} + x_{64} & \geq 1 \quad S = \{2, 5, 6\} \\
x_{30} + x_{31} + x_{32} + x_{40} + x_{42} + x_{46} + x_{50} + x_{51} + x_{56} & \geq 1 \quad S = \{3, 4, 5\} \\
x_{30} + x_{31} + x_{32} + x_{34} + x_{50} + x_{51} + x_{54} + x_{60} + x_{61} + x_{62} & \geq 1 \quad S = \{3, 4, 6\} \\
x_{40} + x_{42} + x_{43} + x_{50} + x_{51} + x_{53} + x_{60} + x_{61} + x_{62} & \geq 1 \quad S = \{4, 5, 6\} \\
x_{10} + x_{15} + x_{16} + x_{20} + x_{26} + x_{30} + x_{35} + x_{40} + x_{45} + x_{46} & \geq 2 \quad S = \{1, 2, 3, 4\} \\
x_{10} + x_{16} + x_{20} + x_{24} + x_{26} + x_{30} + x_{34} + x_{50} + x_{54} + x_{56} & \geq 2 \quad S = \{1, 2, 3, 5\} \\
x_{10} + x_{15} + x_{20} + x_{24} + x_{30} + x_{34} + x_{35} + x_{60} + x_{64} & \geq 2 \quad S = \{1, 2, 3, 6\} \\
x_{10} + x_{13} + x_{16} + x_{20} + x_{23} + x_{26} + x_{40} + x_{43} + x_{46} + x_{50} + x_{53} + x_{56} & \geq 2 \quad S = \{1, 2, 4, 5\} \\
x_{10} + x_{13} + x_{15} + x_{20} + x_{23} + x_{40} + x_{43} + x_{45} + x_{60} + x_{65} & \geq 2 \quad S = \{1, 2, 4, 6\} \\
x_{10} + x_{13} + x_{20} + x_{23} + x_{40} + x_{50} + x_{53} + x_{54} + x_{60} + x_{64} & \geq 2 \quad S = \{1, 2, 5, 6\} \\
x_{10} + x_{12} + x_{16} + x_{30} + x_{32} + x_{40} + x_{42} + x_{46} + x_{50} + x_{56} & \geq 2 \quad S = \{1, 3, 4, 5\} \\
x_{10} + x_{12} + x_{15} + x_{30} + x_{32} + x_{35} + x_{40} + x_{42} + x_{45} + x_{60} + x_{62} + x_{65} & \geq 2 \quad S = \{1, 3, 4, 6\} \\
x_{10} + x_{12} + x_{30} + x_{32} + x_{34} + x_{50} + x_{54} + x_{60} + x_{62} + x_{64} & \geq 2 \quad S = \{1, 3, 5, 6\} \\
x_{10} + x_{12} + x_{13} + x_{40} + x_{42} + x_{43} + x_{50} + x_{53} + x_{60} + x_{62} & \geq 2 \quad S = \{1, 4, 5, 6\} \\
x_{20} + x_{21} + x_{23} + x_{30} + x_{31} + x_{40} + x_{46} + x_{50} + x_{51} + x_{56} & \geq 2 \quad S = \{2, 3, 4, 5\} \\
x_{20} + x_{21} + x_{30} + x_{31} + x_{35} + x_{40} + x_{45} + x_{60} + x_{61} + x_{65} & \geq 2 \quad S = \{2, 3, 4, 6\} \\
x_{20} + x_{21} + x_{23} + x_{40} + x_{43} + x_{50} + x_{51} + x_{60} + x_{61} + x_{64} & \geq 2 \quad S = \{2, 3, 5, 6\} \\
x_{20} + x_{21} + x_{23} + x_{40} + x_{43} + x_{50} + x_{51} + x_{53} + x_{60} + x_{61} + x_{66} & \geq 2 \quad S = \{2, 3, 5, 6\} \\
x_{30} + x_{31} + x_{32} + x_{40} + x_{42} + x_{50} + x_{51} + x_{60} + x_{61} + x_{62} & \geq 2 \quad S = \{3, 4, 5, 6\} \\
x_{10} + x_{16} + x_{20} + x_{26} + x_{30} + x_{40} + x_{46} + x_{50} + x_{56} & \geq 2 \quad S = \{1, 2, 3, 4, 5\} \\
x_{10} + x_{15} + x_{20} + x_{30} + x_{35} + x_{40} + x_{45} + x_{60} + x_{65} & \geq 2 \quad S = \{1, 2, 3, 4, 6\} \\
x_{10} + x_{20} + x_{24} + x_{30} + x_{34} + x_{50} + x_{54} + x_{60} + x_{64} & \geq 2 \quad S = \{1, 2, 3, 5, 6\} \\
x_{10} + x_{13} + x_{20} + x_{23} + x_{40} + x_{43} + x_{50} + x_{53} + x_{60} + x_{62} & \geq 2 \quad S = \{1, 2, 4, 5, 6\} \\
x_{10} + x_{12} + x_{30} + x_{32} + x_{40} + x_{42} + x_{45} + x_{60} + x_{62} + x_{65} & \geq 2 \quad S = \{1, 3, 4, 5, 6\} \\
x_{20} + x_{21} + x_{30} + x_{31} + x_{40} + x_{50} + x_{51} + x_{60} + x_{61} + x_{64} & \geq 2 \quad S = \{2, 3, 4, 5, 6\} \\
\end{align*}\]
\begin{align*}
y_1 & \leq 53 \\
y_2 & \leq 60 \\
y_3 & \leq 39 \\
y_4 & \leq 27 \\
y_5 & \leq 28 \\
y_6 & \leq 25 \\
-\ y_1 + 80x_{01} & \leq 77 \\
-\ y_2 + 80x_{02} & \leq 75 \\
-\ y_3 + 80x_{03} & \leq 69 \\
-\ y_4 + 80x_{04} & \leq 74 \\
-\ y_5 + 80x_{05} & \leq 72 \\
-\ y_6 + 80x_{06} & \leq 72 \\
y_1 - y_2 + 80x_{12} & \leq 51 \\
y_1 - y_3 + 80x_{13} & \leq 59 \\
y_1 - y_5 + 80x_{15} & \leq 43 \\
y_1 - y_6 + 80x_{16} & \leq 48 \\
y_2 - y_1 + 80x_{21} & \leq 47 \\
y_2 - y_3 + 80x_{23} & \leq 49 \\
y_2 - y_4 + 80x_{24} & \leq 50 \\
y_2 - y_6 + 80x_{26} & \leq 51 \\
y_3 - y_1 + 80x_{31} & \leq 66 \\
y_3 - y_2 + 80x_{32} & \leq 59 \\
y_3 - y_4 + 80x_{34} & \leq 50 \\
y_3 - y_5 + 80x_{35} & \leq 62 \\
y_4 - y_2 + 80x_{42} & \leq 61 \\
y_4 - y_3 + 80x_{43} & \leq 54 \\
y_4 - y_5 + 80x_{45} & \leq 55 \\
y_4 - y_6 + 80x_{46} & \leq 70 \\
y_5 - y_1 + 80x_{51} & \leq 49 \\
y_5 - y_3 + 80x_{53} & \leq 62 \\
y_5 - y_4 + 80x_{54} & \leq 50 \\
y_5 - y_6 + 80x_{56} & \leq 51 \\
y_6 - y_1 + 80x_{61} & \leq 39 \\
y_6 - y_2 + 80x_{62} & \leq 47 \\
y_6 - y_4 + 80x_{64} & \leq 52 \\
y_6 - y_5 + 80x_{65} & \leq 37 \\
\end{align*}

\[x_{01}, x_{02}, x_{03}, x_{04}, x_{05}, x_{06}, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{30}, x_{31} \in \{0, 1\}\]

\[x_{32}, x_{34}, x_{35}, x_{40}, x_{42}, x_{43}, x_{45}, x_{46}, x_{50}, x_{51}, x_{53}, x_{54}, x_{56}, x_{60}, x_{61}, x_{62}, x_{64}, x_{65} \in \{0, 1\}\]

2. Find an optimal solution for the vehicle routing problem, using a solver of your choice. Clearly present your solution and optimal objective value.

**Answer:** See Figure 1:
Question 2

Below is a network, with costs indicated on the arcs, and load sizes indicated next to the nodes. Node 0 is the depot. The nodes indicated with circles are origin nodes, where loads have to be picked up. The nodes indicated with squares are destination nodes, where loads have to be delivered. A vehicle can do both deliveries and pickups on a single route, but it has to do all deliveries on the route before any pickups on the route. You have to find the optimal set of vehicle routes that start at the depot and return to the depot, and that serve all the customers. Each vehicle has a capacity of 100 units. Each customer must be served by exactly one vehicle (split deliveries are not allowed).

1. Write down a complete integer linear programming formulation for the vehicle routing problem on the network. With “complete” is meant that you should write the objective function and each individual constraint with the given data substituted in the appropriate places.

\textbf{Answer:} Let \( O = \{1, 2, 3, 4\} \) denote the set of origin (pickup) nodes, and let \( D = \{5, 6, 7, 8\} \) denote the set of destination (delivery) nodes. Let \( N = \{0\} \cup O \cup D \) denote the set of all the nodes.

Convert the given undirected edges into directed arcs in the following way: For each undirected edge \( \{0, i\}, i \in O \cup D \), create two directed arcs, \((0, i)\) and \((i, 0)\), with the same cost as edge \( \{0, i\}\). For each undirected edge \( \{i, j\}, i, j \in O \), create two directed arcs, \((i, j)\) and \((j, i)\), with the same cost as edge \( \{i, j\}\). For each undirected edge \( \{i, j\}\), \(i, j \in D\), create two directed arcs, \((i, j)\) and \((j, i)\), with the same cost as edge \( \{i, j\}\). For each undirected edge \( \{i, j\}\) or \(\{j, i\}\), \(i \in O, j \in D\), create only one directed arc, \((j, i)\), (because deliveries must precede pickups on a route) with the same cost as edge \( \{i, j\}\) or \(\{j, i\}\).

The resulting set of arcs is \( A = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (1, 0), (1, 2), (1, 4), (2, 0), (2, 1), (2, 3), (3, 0), (3, 2), (3, 4), (4, 0), (4, 1), (4, 3), (5, 0), (5, 1), (5, 3), (5, 4), (5, 6), (5, 8), (6, 0), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 7), (7, 0), (7, 2), (7, 3), (7, 6), (7, 8), (8, 0), (8, 1), (8, 2), (8, 3), (8, 4), (8, 5), (8, 7)\}. Let \( G = (N, A) \) denote the resulting directed network where node 0 is the depot.

Let \( c_{ij} \) denote the cost associated with arc \((i, j)\).

Let \( d_i \) denote the quantity to be delivered or picked up at \( i \in O \cup D\).

Note that a subtour cannot include both origin nodes and destination nodes, because there are no arcs from origin nodes to destination nodes. For any subset \( S \) of origin nodes, \( S \subset O \), let \( \alpha(S) \) denote the minimum number of vehicles needed to serve all customers in \( S \). Similarly, for any subset...
$S$ of destination nodes, $S \subset D$, let $\alpha(S)$ denote the minimum number of vehicles needed to serve all customers in $S$.

Assume that if a vehicle visits a node, then it serves that node.

Decision variables:
Let $x_{ij} = \begin{cases} 
1 & \text{if a vehicle travels on arc } (i, j) \in A \\
0 & \text{o/w} 
\end{cases}$

Then the formulation of the problem is as follows:

$$
\begin{align*}
\min & \quad \sum_{(i,j) \in A} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j \in N : (j,i) \in A} x_{ji} = 1 \quad \forall \ i \in O \cup D \\
& \quad \sum_{j \in N : (i,j) \in A} x_{ij} = 1 \quad \forall \ i \in O \cup D \\
& \quad \sum_{(i,j) \in A : i \in S, j \notin S} x_{ij} \geq \alpha(S) \quad \forall \ S \subset O : 2 \leq |S| \leq |O|, \forall \ S \subset D : 2 \leq |S| \leq |D| \\
& \quad x_{ij} \in \{0,1\} \quad \forall \ (i,j) \in A \\
\end{align*}
$$

With the given data, the formulation is

$$
\begin{align*}
\min & \quad \{24x_{01} + 27x_{02} + 28x_{03} + 25x_{04} + 23x_{05} + 26x_{06} + 21x_{07} + 22x_{08} + 24x_{10} \\
& \quad + 31x_{12} + 16x_{14} + 27x_{20} + 31x_{21} + 33x_{23} + 28x_{30} + 33x_{32} + 29x_{34} + 25x_{40} \\
& \quad + 16x_{41} + 29x_{43} + 23x_{50} + 17x_{51} + 26x_{53} + 30x_{54} + 14x_{56} + 17x_{58} + 26x_{60} \\
& \quad + 19x_{61} + 24x_{62} + 25x_{63} + 22x_{64} + 14x_{65} + 19x_{67} + 21x_{70} + 27x_{72} + 21x_{73} \\
& \quad + 19x_{76} + 30x_{78} + 22x_{80} + 20x_{81} + 19x_{82} + 31x_{83} + 32x_{84} + 17x_{85} + 30x_{87}\} \\
\text{s.t.} & \quad x_{10} + x_{12} + x_{14} = 1 \\
& \quad x_{20} + x_{21} + x_{23} = 1 \\
& \quad x_{30} + x_{32} + x_{34} = 1 \\
& \quad x_{40} + x_{41} + x_{43} = 1 \\
& \quad x_{50} + x_{51} + x_{53} + x_{54} + x_{56} + x_{58} = 1 \\
& \quad x_{60} + x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{67} = 1 \\
& \quad x_{70} + x_{72} + x_{73} + x_{76} + x_{78} = 1 \\
& \quad x_{80} + x_{81} + x_{82} + x_{83} + x_{84} + x_{85} + x_{87} = 1 \\
& \quad x_{90} + x_{91} + x_{93} + x_{94} + x_{95} + x_{96} + x_{98} = 1 \\
& \quad x_{10} + x_{12} + x_{14} + x_{20} + x_{23} \geq 2 \quad S = \{1, 2\} \\
& \quad x_{10} + x_{12} + x_{14} + x_{30} + x_{32} + x_{34} \geq 1 \quad S = \{1, 3\} \\
& \quad x_{10} + x_{12} + x_{40} + x_{43} \geq 1 \quad S = \{1, 4\} \\
& \quad x_{20} + x_{21} + x_{30} + x_{34} \geq 1 \quad S = \{2, 3\} \\
& \quad x_{20} + x_{21} + x_{23} + x_{40} + x_{41} \geq 2 \quad S = \{2, 4\} \\
& \quad x_{30} + x_{32} + x_{40} + x_{41} \geq 1 \quad S = \{3, 4\} \\
& \quad x_{10} + x_{14} + x_{20} + x_{30} + x_{34} \geq 2 \quad S = \{1, 2, 3\}
\end{align*}
$$
2. Find an optimal solution for the vehicle routing problem, using a solver of your choice. Clearly present your solution and optimal objective value.

**Answer:** See Figure 2:
Figure 2: Total Cost = 220