Name

Please be neat and show all your work so that I can give you partial credit. GOOD LUCK.

Question 1
Question 2
Question 3
Question 4
Total
1. a. (10) Show that the period of a state in an irreducible Markov chain with \( N \) states is less than or equal to \( N \).

b. (10) Show that there are no null recurrent states in a Markov chain with finite state space.

c. (10) Show that not all states in a Markov chain with finite state space are transient.
2. (25) Let $X_n$ be the sum of the first $n$ outcomes of tossing of a six sided fair die repeatedly in an independent fashion. Compute

$$\lim_{n \to \infty} P\{X_n \text{ is divisible by } 7\}.$$
3. a. (10) Consider an ordinary renewal process \( \{N(t) : t \geq 0\} \) with inter-renewal time distribution \( F \). Compute \( E[(N(t))^2] \).

b. (15) Now suppose that \( \{N(t) : t \geq 0\} \) is a delayed renewal process. Let \( G \) be the distribution of \( X_1 \) and let \( F \) denote the distribution of \( X_n, n \geq 2 \). Compute \( E[(N(t))^2] \).
4. a. (10) Let \( \{(X_n, R_n, C_n) : n \geq 1\} \) be a sequence of independent identically distributed random variables, with \( X_n \) being non-negative for all \( n \geq 1 \). Let \( \{Z_1(t) : t \geq 0\} \) be the renewal reward process generated by \( \{(X_n, R_n) : n \geq 1\} \) and let \( \{Z_2(t) : t \geq 0\} \) be the renewal reward process generated by \( \{(X_n, C_n) : n \geq 1\} \). Suppose \( E[X_1] < \infty, E[R_1] < \infty \) and \( E[C_1] < \infty \). Compute

\[
\lim_{t \to \infty} \frac{Z_1(t)}{Z_2(t)}.
\]

b. (10) Let \( X \) be a non-negative random variable with distribution function \( F \) and let \( X_e \) be the random variable with distribution function \( F_e \) where \( F_e(x) = \int_0^x (1 - F(y))dy/E[X] \). Thus, \( X_e \) has the equilibrium distribution. Obtain an expression for \( E[X_e^r] \) for \( r > 0 \) in terms of the moments of \( X \).