Lecture 18: Gaussian Channel

- Gaussian channel
- Gaussian channel capacity
Mona Lisa in AWGN

Mona Lisa

Noisy Mona Lisa
**Gaussian channel**

- the most important continuous alphabet channel: AWGN
- $Y_i = X_i + Z_i$, noise $Z_i \sim \mathcal{N}(0, N)$, independent of $X_i$
- model for communication channels: satellite links, wireless phone
Channel capacity of AWGN

- intuition: \( C = \log \) number of distinguishable inputs

- if \( N = 0 \), \( C = \infty \)

- if no power constraint on the input, \( C = \infty \)

- to make it more meaningful, impose average power constraint: for any codewords \((x_1, \ldots, x_n)\)
  \[
  \frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P
  \]
Naive way of using Gaussian channel

- Binary phase-shift keying (BPSK)
- transmit 1 bit over the channel
- $1 \rightarrow +\sqrt{P}$, $0 \rightarrow -\sqrt{P}$
- $Y = \pm \sqrt{P} + Z$
- Probability of error
  
  $$P_e = 1 - \Phi(\sqrt{P/N})$$

  normal cumulative probability function (CDF):
  $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

- convert Gaussian channel into a discrete BSC with $p = P_e$. Lose information in quantization
Definition: Gaussian channel capacity

- $C = \max_{f(x): EX^2 \leq P} I(X; Y)$
- we can calculate from here
  $$C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right)$$
  maximum attained when $X \sim \mathcal{N}(0, P)$
$C$ as maximum data rate

- we can also show this $C$ is the supremum of rate achievable for AWGN

- definition: a rate $R$ is achievable for Gaussian channel with power constraint $P$:

  if there exists a $(2^{nR}, n)$ codes with maximum probability of error $\lambda^n = \max_{i=1}^{2^{nR}} \lambda_i \to 0$ as $n \to \infty$. 
Sphere packing

why we may construct \((2^nC, n)\) codes with a small probability of error?

Fix one codeword

- consider any codeword of length \(n\)
- received vector \(\sim \mathcal{N}(\text{true codeword}, N)\)
- with high probability, received vector contained in a sphere of radius \(\sqrt{n(N + \epsilon)}\) around true codeword
- assign each ball a codeword
Consider all codewords

- with power constraint, with high probability the space of received vector is a sphere with radius $\sqrt{n(P + N)}$

- volume of $n$-dimensional sphere $= C_n r^n$ for constant $C_n$ and radius $r_n$

- how many codewords can we pack in a “total power sphere”?

$$\frac{C_n (n(P + N))^{n/2}}{C_n (nN)^{n/2}} = \left(1 + \frac{P}{N}\right)^{n/2}$$

- rate of this codebook $= \log_2(\text{size of the codewords})/n$:

$$= \frac{1}{2} \log_2 \left(1 + \frac{P}{N}\right)$$
sphere packing
Gaussian channel capacity theorem

**Theorem.** The capacity of a Gaussian channel with power constraint $P$ and noise variance $N$ is

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right) \text{ bits per transmission}$$

Proof: 1) achievability; 2) converse
New stuff in proof

Achievability:

• codeword elements generated i.i.d. according $X_j(i) \sim \mathcal{N}(0, P - \epsilon)$. So

$$\frac{1}{n} X_i^2 \to P - \epsilon$$

• power outage error

$$E_0 = \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j^2(1) > P \right\}$$

small according to law of large number

Converse: Gaussian distribution has maximum entropy
Band-limited channels

- more realistic channel model: band-limited continuous AWGN

\[ Y(t) = (X(t) + Z(t)) * h(t) \]

*: convolution

- Nyquist-Shannon sampling theorem:

  sampling a band-limited signal at sampling rate \( 1/(2W) \) is sufficient to reconstruct the signal from samples

  “two samples per period”
Capacity of continuous-time band-limited AWGN

- noise has power spectral density $N_0/2$ watts/hertz, bandwidth $W$ hertz, noise power = $N_0W$
- signal power $P$ watts
- $2W$ samples each second
- channel capacity

$$C = W \log \left( 1 + \frac{P}{N_0W} \right) \text{ bits per second}$$

- when $W \to \infty$, $C \to \frac{P}{N_0} \log_2 e \text{ bits per second}$
Telephone line

- telephone signal are band-limited to 3300 Hz
- SNR = 33 dB: $P/(N_0W) = 2000$
- capacity = 36 kb/s
- practical modems achieve transmission rates up to 33.6 kb/s uplink and downlink
- ADSL achieves 56 kb/s downlink (asymmetric data rate)
Summary

• additive white Gaussian noise (AWGN) channel

• noise power: $N$, signal power constraint $P$, capacity

\[ C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) \]

• band-limited channel with bandwidth $= W$

\[ C = W \log \left( 1 + \frac{P}{N_0 W} \right) \]