Optimal "Hit Covering" and "Part Assignment" in Distributed Inventory Systems*

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Abstract

This paper addresses some problems arisisng in the real-time management of expensive-item, low-volume, distributed inventory systems. Our analysis is motivated by the management of the so-called, *key repairable* part inventory, in contemporary airline industry. The high cost of these parts, combined with their criticality for flight safety, make their effective and efficient provision to the different stations, a key competitive factor for the airline companies. On the other hand, the stochasticity of the system demand, combined with the uneconomical character of any policy based on over-stocking, result in significant exposure of the system stations to under-stocking and part shortages. It is common industry practice that a part shortage ("hit") at a certain station is covered by the inter-shipment of one unit from another station currently owning the part. The first part of this paper rationalizes the selection of the hit-covering station, by proposing a number of decision-making criteria, and analyzing the properties of the resulting policies. The second part of the paper addresses the complementary problem of distributing an arriving batch of new part units among a number of under-stocked stations. It is shown that the "one-unit part assignment" problem can be reduced to an equivalent "hit-covering" problem, while the "n-unit part assignment" problem can be solved optimally through a *greedy* algorithm that formulates and solves a succession of n "one-unit part-assignment" subproblems.

1 Introduction

The research presented in this paper has been motivated by the maintenance operations taking place in the airline industries. Specifically, safety considerations require that some crucial parts of the

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planes are fully functional before a flight departs. As a result, considerable amount of unscheduled maintenance takes place at the different (major) airports covered by the airline network; in the airline industry jargon, these airports are called *stations*.

To meet the needs of this type of maintenance without experiencing long flight delays, some amount of inventory must be maintained at the different stations. On the other hand, most of the parts involved are expensive and/or bulky items, and therefore, it is imperative to maintain at a minimum the amount of inventory kept at each station. To give an idea of the cost range of these items, we mention that most of them are classified as *key repairable* or *rotable* items, i.e., whenever they fail they are not discarded, but they are repaired and added to the stock of spare parts. In fact, most of the spare stock comes from such repaired units.

To the extent that the station inventories are replenished from the company's central repair shop, and parts are transported through the regular flight schedule, *order* costs are negligible. Hence, ideally, the optimal inventory to be kept at each station should be determined from an effort to balance the inventory *holding* costs versus the costs arising from part *shortages*. It turns out, however, that the evaluation of the "optimal" inventory level to be kept at each station is a rather difficult problem, primarily due to the intangible nature of the underlying shortage costs, and the potential non-stationary nature of the demand experienced at the different stations (e.g., due to changes in flight schedules, weather conditions, etc.). An introduction to the problem of evaluating the optimal inventory level at the different stations, based on the scheduled flight activity, can be found in [2]. In this paper, we consider the "real-time" operation of the system, and design policies for dealing with situations like stock-outs, and partial/incremental part replenishment.

More specifically, the stochasticity of the station demands, combined with the uneconomical nature of any solution based on over-stocking, result in significant *exposure* of the system stations, i.e., situations in which the station has a pending flight needing a part, while, both, itself and the repair shop are out of stock. The occurrence of such an event is characterized as a *"hit"* in the industry jargon. It is common practice in the airline industry that the demand of a hit is met by another station which has the part available. Selecting the station to cover the part need arising by a hit is an economically crucial decision in the inventory management of contemporary airlines. The resulting problem is characterized as the *"hit covering"* problem.

Obviously, the criticality of flight delays implies that stations which are good candidates to draw from, are those that can ship the part fast. However, (i) the stochasticity inherent in airline flight schedules with respect to arrival and departure times, and (ii) the strong nonlinearity of shortage costs with respect to the experienced delays, make time-based selection a rather fuzzy criterion. Furthermore, there is always the danger that "blind" withdrawal of a unit from a certain station might leave that station over-exposed to the possibility of another hit. Hence, it seems most appropriate that information on shipping times should be mainly used in a *pre-selection stage*, in order to eliminate "bad" candidate stations, i.e., stations that would cover the hit with extremely long delays.

The final selection among the remaining stations must be based on part *availability* considerations. Specifically, we would like to select the station that will provide the required part in a way that the potential disruption due to additional hits over the system of candidate stations is minimal. Two analytical criteria able to quantify this last objective are: (i) *the maximization of the probability of taking no hit over the system of candidate stations*, and (ii) *the minimization of the number of hits expected to occur in this system.*

Given that every hit is eventually covered by inter-shipping among the system stations, we believe that the first of the above criteria is more appropriate for the airline industry context. We consider both criteria in this analysis, since we want also to study the extent of their equivalence. It turns out that both of these criteria rationalize effectively the different factors weighing in the selection process. However, they are not exactly equivalent, since, as we shall show, it is possible to identify cases in which each criterion leads to a different selection.

A problem complementary to "hit covering" is that of the *"part assignment"*. This problem arises whenever a new batch of spare units is introduced into the system – either through procurement or repairing – while more than one station have pending requests for part replenishment. Assuming that there are no *batch-sizing* restrictions – an assumption justified by the negligible freight costs and the dense flight schedules of the airline companies – the selection of the station(s) to receive these parts is driven by the same "quality-of-service" criteria that were proposed for the "hit covering" problem. In fact, there is strong relationship between the "hit covering" and the "part assignment" problem: The "one-unit part assignment" problem is decomposeable to a series of two-station subproblems, each of which can be reduced to a two-station "hit covering" problem, while the "n-unit part assignment" problem can be solved optimally through a greedy alogirthm that formulates and solves a succession of n "one-unit part assignment" subproblems. Hence, by initially addressing the "hit covering" problem, we also set the stage for the analysis of the optimal "part assignment".

An important factor that influences the optimal decision of, both, the "hit covering" and the "part assignment" problems, is the *time horizon* over which the considered system of stations will operate under the inventory status resulting from the policy action. Typically, this will be the time until a new replenishment is scheduled. The length of this horizon also characterizes, to a large extent, the nature of the processes driving the demand at the different stations. In particular, the rather short length of this time period allows us to assume the stability of the flight schedule, and model the demand occurring at each station as a *Poisson* process, with a *constant* arrival rate, determined by the volume of flights scheduled to the station over that period, when adjusted to account for the extent of the maintenance activity typically taking place at that station. This demand model is also corroborated in [2]. An intuitive argument that explains the Poisson nature of the station demands, under the above assumptions, can be based on the fact that the random variable characterizing the life of a certain part, when considered over the entire fleet, is quite broadly dispersed. On the other hand, the routing of each plane to the different stations follows a rather random pattern. Hence, each arrival at a station is an equiprobable candidate for part replacement.

The rest of the paper is structured as follows: Section 2 provides the detailed formulation of the "hit covering" problem according to both selection criteria, and establishes the decomposability of these formulations to a series of two-station subproblems. Section 3 exploits this decomposability in order to analyze the optimal solution of the "hit covering" problem, when it is formulated according to the first selection criterion discussed above. Section 4 performs the same analysis of the "hit covering" problem, when the second selection criterion is used. Furthermore, Section 4 investigates the equivalence of the two criteria with respect to the optimal selection. Section 5 formulates the "one-unit part assignment" problem, establishes its reducibility to the "hit covering" problem, and subsequently, exploits this reducibility in order to derive properties of the optimal solution. Section 6 studies the "n-unit part assignment" problem, and shows that it can be solved optimally through a greedy algorithm that solves a succession of n "one-unit part assigment" subproblems. Finally, Section 7 draws conclusions, and suggests directions for future research.

2 The "Hit Covering" Problem

An analytical formulation of the *"hit covering"* problem is as follows:

We are given a distributed one-part inventory system of $N+1$ stations, $S_1, \ldots, S_N, S_{N+1}$. Station S_i, $i = 1, \ldots, N$, is currently holding I_i units of inventory, and it faces a local demand D_i, which is *Poisson* distributed, with rate λ_i. Furthermore, demand processes at different stations are independent of each other. On the other hand, station S_{N+1} is experiencing a "hit", i.e., it needs immediately one unit of the part, while itself, as well as the central warehouse, is out of stock. Furthermore, no replenishment is scheduled until some time T. Therefore, we would like to meet the immediate demand at station S_{N+1} by shipping to it a part from one of the stations $S_1, \ldots, S_N$. However, the shipping station must be selected in such a way that the *"exposure" of the subsystem $\mathcal{S}$ of stations $S_1, \ldots, S_N$ is minimized*. This last requirement can be quantified as follows:

Let $I_i^{(j)}$, $i = 1, \ldots, N$, denote the inventory status at stations S_i, $i = 1, \ldots, N$, after the required part has been shipped to station S_{N+1} from station S_j, $j = 1, \ldots, N$. Then,

$$I_i^{(j)} = \begin{cases} I_i - 1, & \text{if } i = j \\ I_i, & \text{o.w.} \end{cases} = I_i - \delta_{ij} \tag{1}$$

where δ_{ij} is *Kronecker's* delta. The probability that station S_i, $i = 1, \ldots, N$, will not experience a "hit" (shortage) over period T, is given by:

$$\begin{aligned} P_{NH}(i|j) &= P[\text{No hit at station } S_i \text{ over } T\text{, after drawing one part from station } S_j] \\ &= P[D_i(T) \leq I_i^{(j)}] = \sum_{k=0}^{I_i^{(j)}} \frac{(\lambda_i T)^k e^{-\lambda_i T}}{k!} \end{aligned} \tag{2}$$

Since the demand experienced by each station is independently driven, the probability that the entire system will experience no hit is given by:

$$\begin{aligned} P_{NH}(\mathcal{S}|j) &= P[\text{no hit in system } \mathcal{S} \text{ over period } T\text{, after drawing one part from station } S_j] \\ &= \prod_{i=1}^{N} P_{NH}(i|j) \end{aligned} \tag{3}$$

Hence, the first of the two selection criteria introduced in the previous section – i.e., seeking to select the station $S_{i^{*(1)}}$ that will provide the required part, by trying to maximize the probability of system-wise safety – is formally expressed by:

$$i^{*(1)} = \arg\max_j P_{NH}(\mathcal{S}|j) \tag{4}$$

The alternative criterion of minimizing the expected number of "hits" in the entire system $\mathcal{S}$, over period T, is formalized as follows: Let H_i, $i = 1, \ldots, N$, denote the number of hits at station S_i over the considered period T. Also, define $H_{\mathcal{S}} = \sum_{i=1}^{N} H_i$, the number of "hits" received by the entire system over period T. Then:

$$\begin{aligned} E[H_{\mathcal{S}}|j] &= \sum_{i=1}^{N} E[H_i|j] \\ &= \sum_{i=1}^{N} \sum_{k=1}^{\infty} k P[D_i = k + I_i^{(j)}] = \sum_{i=1}^{N} \sum_{k=1}^{\infty} k \frac{(\lambda_i T)^{k+I_i^{(j)}} e^{-\lambda_i T}}{(k + I_i^{(j)})!} \end{aligned} \tag{5}$$

and the optimal selection of the part-shipping station $S_{i^{*(2)}}$ is:

$$i^{*(2)} = \arg\min_j E[H_{\mathcal{S}}|j] \tag{6}$$

In the following two sections we study the optimization problems defined by Equations 4 and 6, respectively, and we derive properties of their solution. It is interesting to notice, however, that the multiplicative nature of criterion (1) with respect to the system components, implies that any solution $i^{*(1)}$ will be optimal for the entire system $\mathcal{S}$, if and only if it is optimal for every subsystem $\{S_{i^{*(1)}}, S_j\}$, $j = 1, \ldots, N, j \neq i^{*(1)}$. A similar remark arises from the additive nature of criterion (2). Hence, *in order to analyze the properties of the solution of these problems, it is sufficient to study the problem restriction for* $N = 2$. Definitely, this approach increases the tractability of the analysis, and it is adopted in the sequel.

3 Maximizing the Probability of "No Hit, System-wise"

Following the remark in the last paragraph of the previous section, here we concentrate on the case where $N = 2$, i.e., there are two stations that can provide the part. Let us denote these stations by S_1, S_2, their demand rates by $\lambda_1, \lambda_2 > 0$, and their inventory levels by $I_1, I_2 \geq 1$, respectively. Furthermore, without loss of generality, let us assume that:

$$\lambda_1 \geq \lambda_2 \tag{7}$$

A restatement of Equation 4 in this case is as follows:

$$\begin{aligned}
1 = \arg\max_{j=1,2} P_{NH}(\mathcal{S}|j) \quad &\Longleftrightarrow \\
P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2) \geq 0 \quad &\Longleftrightarrow \\
\left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^k e^{-\lambda_1 T}}{k!}\right]\left[\sum_{k=0}^{I_2} \frac{(\lambda_2 T)^k e^{-\lambda_2 T}}{k!}\right] - \left[\sum_{k=0}^{I_1} \frac{(\lambda_1 T)^k e^{-\lambda_1 T}}{k!}\right]\left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^k e^{-\lambda_2 T}}{k!}\right] \geq 0 \quad &\Longleftrightarrow \\
\left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^k e^{-\lambda_1 T}}{k!}\right] \frac{(\lambda_2 T)^{I_2} e^{-\lambda_2 T}}{I_2!} - \left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^k e^{-\lambda_2 T}}{k!}\right] \frac{(\lambda_1 T)^{I_1} e^{-\lambda_1 T}}{I_1!} \geq 0 \quad &\Longleftrightarrow \\
P[D_1(T) \leq I_1 - 1]P[D_2(T) = I_2] - P[D_2(T) \leq I_2 - 1]P[D_1(T) = I_1] \geq 0 \quad &\Longleftrightarrow \\
\frac{P[D_1(T) \leq I_1 - 1]}{P[D_2(T) \leq I_2 - 1]} \geq \frac{P[D_1(T) = I_1]}{P[D_2(T) = I_2]} &
\end{aligned} \tag{8}$$

3.1 Case 1: $I_2 \geq I_1$

In this section we show that when $\lambda_1 \geq \lambda_2$ and $I_2 \geq I_1$, Equation 8 implies that it is always optimal to draw the required part from the inventory of station S_2. This result is rather intuitive since, under the specified assumptions, station S_2 can be considered as over-stocked with respect to station S_1. Its formal statement and proof are as follows:

Theorem 1 *For the two-station "hit covering" problem, formulated according to the criterion of Equation 4, and with*

1. $\lambda_1 \geq \lambda_2$,

2. $I_2 \geq I_1$,

it is always optimal to cover the hit by drawing a part from station S_2.

Proof: From Equation 8, we have

$$\begin{aligned} P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2) &= \\ \left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^k e^{-\lambda_1 T}}{k!}\right] \frac{(\lambda_2 T)^{I_2} e^{-\lambda_2 T}}{I_2!} - \left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^k e^{-\lambda_2 T}}{k!}\right] \frac{(\lambda_1 T)^{I_1} e^{-\lambda_1 T}}{I_1!} &= \\ \left\{\left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^k}{k!}\right] \frac{(\lambda_2 T)^{I_2}}{I_2!} - \left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^k}{k!}\right] \frac{(\lambda_1 T)^{I_1}}{I_1!}\right\} e^{-(\lambda_1+\lambda_2)T} &= \\ \left\{\left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^{k-I_1}}{k! I_2!}\right] - \left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^{k-I_2}}{k! I_1!}\right]\right\} (\lambda_1 T)^{I_1} (\lambda_2 T)^{I_2} e^{-(\lambda_1+\lambda_2)T} & \end{aligned} \tag{9}$$

Therefore, for $T \neq 0$,

$$\begin{aligned} \operatorname{sgn}\left(P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2)\right) &= \\ \operatorname{sgn}\left(\left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^{k-I_1}}{k! I_2!}\right] - \left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^{k-I_2}}{k! I_1!}\right]\right) & \end{aligned} \tag{10}$$

But

$$\begin{aligned} \left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^{k-I_1}}{k! I_2!}\right] - \left[\sum_{k=0}^{I_2-1} \frac{(\lambda_2 T)^{k-I_2}}{k! I_1!}\right] &\leq \\ \left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^{k-I_1}}{k! I_2!}\right] - \left[\sum_{k=I_2-I_1}^{I_2-1} \frac{(\lambda_2 T)^{k-I_2}}{k! I_1!}\right] &= \\ \left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^{k-I_1}}{k! I_2!}\right] - \left[\sum_{k=0}^{I_1-1} \frac{(\lambda_2 T)^{k-I_1}}{(k+I_2-I_1)! I_1!}\right] & \end{aligned} \tag{11}$$

For $0 \leq k \leq I_1 - 1$,

$$\lambda_1 \geq \lambda_2 \iff$$

$$
\begin{aligned}
\lambda_1^{I_1-k} \geq \lambda_2^{I_1-k} \quad &\Longleftrightarrow \\
\lambda_1^{k-I_1} \leq \lambda_2^{k-I_1} \quad &\Longleftrightarrow \\
(\lambda_1 T)^{k-I_1} \leq (\lambda_2 T)^{k-I_1} & \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{(k+I_2-I_1)!}{k!} \leq \frac{I_2!}{I_1!} \quad &\Longleftrightarrow \\
\frac{1}{k!I_2!} \leq \frac{1}{(k+I_2-I_1)!I_1!} & \qquad (13)
\end{aligned}
$$

Hence, from Equations 12 and 13, we get:

$$
\left[\sum_{k=0}^{I_1-1} \frac{(\lambda_1 T)^{k-I_1}}{k!I_2!}\right] \leq \left[\sum_{k=0}^{I_1-1} \frac{(\lambda_2 T)^{k-I_1}}{(k+I_2-I_1)!I_1!}\right] \qquad (14)
$$

When $T = 0$, the last line of Equation 9 implies that:

$$
P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2) = 0 \qquad (15)
$$

Equations 8, 10, 11, 14 and 15 together imply that, under the stated assumptions,

$$
\arg \max_{j=1,2} P_{NH}(\mathcal{S}|j) = 2 \qquad (16)
$$

□

The result of Theorem 1 is of special interest as it covers the case where $I_1 = I_2 = \ldots = I_N = 1$, a situation arising quite often in the airline industry. It is straightforward to see that in this case, the optimal selection is a station corresponding to the lowest demand rate. We state this result as a corollary:

Corollary 1 *In case that* $I_i = 1, \ i = 1, \ldots, N$,

$$
i^{*(1)} = \arg \max_j P_{NH}(\mathcal{S}|j) = \arg \min_j \lambda_j \qquad (17)
$$

3.2 Case 2: $I_1 > I_2$

Next, we investigate the case where $I_1 > I_2$. As we shall see, in this case there is no "clear-cut" dominance of any of the two stations. In other words, both stations, S_1 and S_2, might be selected as the shipping station, depending on the specific values of $\lambda_1, \lambda_2, I_1, I_2$. We establish this result by means of some examples.

Example 1: Station S_2 is the best candidate First, consider the case where $\lambda_1 = 5$, $\lambda_2 = 2$ parts/day, and $I_1 = 2$, $I_2 = 1$ parts. Also, let the considered planning horizon be one day, i.e., $T = 1$. Then, from the fourth line of Equation 8:

$$\begin{aligned} P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2) &= \\ \left[\sum_{k=0}^{1} \frac{5^k e^{-5}}{k!}\right] \frac{2^1 e^{-2}}{1!} - \left[\sum_{k=0}^{0} \frac{2^k e^{-2}}{k!}\right] \frac{5^2 e^{-5}}{2!} &= -4.56 \times 10^{-4} \end{aligned} \tag{18}$$

Therefore, station S_2 is to be chosen as the providing station. □

Example 2: Station S_1 is the best candidate On the other hand, if, in the previous example, we increase the inventory of station S_1 by one unit, i.e., $\lambda_1 = 5$, $\lambda_2 = 2$ parts/ day, $I_1 = 3$, $I_2 = 1$ parts, and $T = 1$ day, then:

$$\begin{aligned} P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2) &= \\ \left[\sum_{k=0}^{2} \frac{5^k e^{-5}}{k!}\right] \frac{2^1 e^{-2}}{1!} - \left[\sum_{k=0}^{0} \frac{2^k e^{-2}}{k!}\right] \frac{5^3 e^{-5}}{3!} &\simeq 0.015 \end{aligned} \tag{19}$$

Hence, we must draw a part from station S_1. □

3.3 The dependency of the optimal decision on parameter T

When $I_1 > I_2$, the length of the time horizon T is also a decisive factor for the optimal selection, although it need not always be the case. We demonstrate this effect with the following examples:

Example 3: Optimal decision depends on the length of the planning horizon, T Let us consider the scenario of Example 1, but with the length of the planning horizon T varying over the interval $[0, \infty)$. Then, from the third line of Equation 9, we get:

$$\begin{aligned} \operatorname{sgn}\left(P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2)\right) &= \\ \operatorname{sgn}\left(\left[\sum_{k=0}^{1} \frac{(5T)^k}{k!}\right] \frac{(2T)^1}{1!} - \left[\sum_{k=0}^{0} \frac{(2T)^k}{k!}\right] \frac{(5T)^2}{2!}\right) &= \\ \operatorname{sgn}\left(T(2 - 2.5T)\right) &= \begin{cases} 0, & \text{for } T = 0 \text{ or } T = 0.8 \\ 1, & \text{when } 0 < T < 0.8 \\ -1, & \text{when } T > 0.8 \end{cases} \end{aligned} \tag{20}$$

Hence, the optimal selection is:

- station S_1, if the planning horizon T is in the interval $(0, 0.8)$,

- station S_2, if the planning horizon is $T > 0.8$,
- and it can be arbitrary for $T = 0$ or $T = 0.8$.

□

Example 4: Optimal decision is not affected from the length of the planning horizon, T The analysis of Example 3 with respect to the effect the length of planning horizon T, when carried out on the data of Example 2, gives the following result:

$$\begin{aligned} \text{sgn}\left(P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2)\right) &= \\ \text{sgn}\left(\left[\sum_{k=0}^{2}\frac{(5T)^k}{k!}\right]\frac{(2T)^1}{1!} - \left[\sum_{k=0}^{0}\frac{(2T)^k}{k!}\right]\frac{(5T)^3}{3!}\right) &= \\ \text{sgn}\left(T(2+10T+\frac{25}{6}T^2)\right) &= \begin{cases} 0, & \text{for } T = 0 \\ 1, & \text{when } T > 0 \end{cases} \end{aligned} \tag{21}$$

Hence, the policy that draws a part from station S_1 is optimal for any planning horizon. □

Generalizing Examples 3 and 4, we can see that the optimal selection policy is structured with respect to T, for a given set of parameters $\lambda_1,\ \lambda_2,\ I_1,\ I_2$, by means of the *real* roots of the polynomial:

$$\Lambda(T) \equiv \left[\sum_{k=0}^{I_1-1}\frac{(\lambda_1 T)^k}{k!}\right]\frac{(\lambda_2 T)^{I_2}}{I_2!} - \left[\sum_{k=0}^{I_2-1}\frac{(\lambda_2 T)^k}{k!}\right]\frac{(\lambda_1 T)^{I_1}}{I_1!} \tag{22}$$

We conclude this analysis of the dependence of the optimal decision on the length of the planning horizon, by studying the asymptotic behavior of the optimal policy as $T \to 0$, and $T \to \infty$. It turns out that for both these cases, the problem of selecting between stations S_1 and S_2 becomes irrelevant, as the difference of the objective values (i.e., probability of "no hit, system-wise") resulting by the two selections converges to zero. The first of these results can be interpreted intuitively by the *"memoryless"* property of the Poisson distribution, and the second from the fact that the expected demand at each station is a *strictly increasing* function of time. We formally state and prove these results in the next theorem and its corollary.

Theorem 2 *For the two-station "hit covering" problem,*

1. $\lim_{T\to 0}\{P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2)\} = 0$
2. $\lim_{T\to \infty}\{P_{NH}(\mathcal{S}|1) - P_{NH}(\mathcal{S}|2)\} = 0$

Proof: From the first and third rows of Equation 9:

$$\begin{aligned}\lim_{T\to 0,\infty}\{P_{NH}(\mathcal{S}|1)-P_{NH}(\mathcal{S}|2)\} &= \\ \lim_{T\to 0,\infty}\left\{\left\{\left[\sum_{k=0}^{I_1-1}\frac{(\lambda_1 T)^k}{k!}\right]\frac{(\lambda_2 T)^{I_2}}{I_2!}-\left[\sum_{k=0}^{I_2-1}\frac{(\lambda_2 T)^k}{k!}\right]\frac{(\lambda_1 T)^{I_1}}{I_1!}\right\}e^{-(\lambda_1+\lambda_2)T}\right\} &= \\ \lim_{T\to 0,\infty}\frac{\Lambda(T)}{e^{(\lambda_1+\lambda_2)T}} & \qquad (23)\end{aligned}$$

Then,

$$\lim_{T\to 0}\{P_{NH}(\mathcal{S}|1)-P_{NH}(\mathcal{S}|2)\}=\lim_{T\to 0}\frac{\Lambda(T)}{e^{(\lambda_1+\lambda_2)T}}=\frac{\lim_{T\to 0}\Lambda(T)}{\lim_{T\to 0}e^{(\lambda_1+\lambda_2)T}}=\lim_{T\to 0}\Lambda(T)=0 \qquad (24)$$

The last equality in Equation 24 results from the definition of $\Lambda(T)$ (cf. Equation 22), and the fact that $I_1, I_2 > 0$ (since stations S_1 and S_2 are candidate stations for providing the part).

The computation of $\lim_{T\to\infty}\frac{\Lambda(T)}{e^{(\lambda_1+\lambda_2)T}}$ is more involved, since taking the limits term-wise gives rise to the indefinite form ∞/∞, but the result can be derived by (repetitive) application of L'Hospital's rule (cf. e.g., [1]). □

Corollary 2 *As $T\to 0,\infty$, any station S_j that can provide the part in due time, is an optimal selection for the N-station "hit covering" problem, formulated with respect to the* first *criterion, i.e.,*

$$i^{*(1)}=\arg\max_j P_{NH}(\mathcal{S}|j)=j, \quad \forall j\in\{1,\ldots,N\} \qquad (25)$$

Proof: The result of Corollary 2 is an immediate consequence of Theorem 2, and the concluding remark of Section 2, regarding the pairwise decomposability of problem, with respect to the candidate stations. □

Finally, it is interesting to notice that a line of analysis similar to that undertaken above, establishes that:

$$\lim_{\lambda_i\to\infty}\{P_{NH}(\mathcal{S}|1)-P_{NH}(\mathcal{S}|2)\}=\lim_{\lambda_i\to\infty}\frac{\Lambda(T)}{e^{(\lambda_1+\lambda_2)T}}=0, \quad \forall i=1,2 \qquad (26)$$

This result expresses *the infinitesimal marginal value of inventory at a certain station, in case of excessive demand!*

4 Minimizing the Expected Number of Hits

In this section we study the "hit covering" problem under the selection criterion expressed by Equation 6, i.e., when we are trying to minimize the expected number of "hits" taken by the entire system, over the considered planning horizon T. As it was remarked at the end of Section 2, the additive structure of the considered objective function with respect to the different stations allows us to restrict our analysis to the case of a two-station system. It turns out that the optimal policy under this new criterion presents similar properties to the policy developed in the previous section. Specifically, for a two-station system with $\lambda_1 \geq \lambda_2$ and $I_2 \geq I_1$, this new criterion is also optimized by selecting the relatively over-stocked station S_2 as the shipping station. Also, for the complementary case $I_1 > I_2$, any of the two stations can be the optimal selection, depending on the specific values of λ_1, λ_2, I_1 and I_2.

An additional question that arises naturally, concerns the *equivalence* of the two selection criteria, i.e., whether optimality of a station with respect to one of them implies the optimality with respect to the other. From the above discussion, it is obvious that they are indeed equivalent in the case where $\lambda_1 \geq \lambda_2$ and $I_2 \geq I_1$. However, we show below that when $I_1 > I_2$, it is possible to identify two-station problem cases where application of each criterion leads to a different selection. Furthermore, we were able to identify a subtler relationship between the two criteria, which admits an interesting intuitive interpretation, and it can rationalize the selection between them. This result and its implications in the airline industry context are the content of Theorem 6 and its ensuing discussion.

Similar to the analysis of Section 3, in order to proceed with the formal development of the section results, we need a closed-form expression for the difference $E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2]$, arising in the two-station problem. We derive it below.

Lemma 1 *Let D denote a Poisson-distributed random variable with rate λ. Then,*

$$\forall\ k \in \mathcal{N}^+, \qquad \lambda \cdot P[D = k-1] - k \cdot P[D = k] = 0 \tag{27}$$

Proof: Since D is Poisson distributed,

$$\begin{aligned} \lambda \cdot P[D = k-1] - k \cdot P[D = k] &= \lambda \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} - k \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \frac{\lambda^k e^{-\lambda}}{(k-1)!} - \frac{\lambda^k e^{-\lambda}}{(k-1)!} \\ &= 0 \end{aligned} \tag{28}$$

□

Lemma 2 *Given a station S_j with Poisson-distributed demand D_j of rate λ_j, and available inventory*

I_j*, the expected number of hits over a time horizon* T *is given by:*

$$E[H_j|\lambda_j, I_j, T] = (\lambda_j T) \cdot P[D_j(T) > I_j - 1] - I_j \cdot P[D_j(T) > I_j] \tag{29}$$

Proof: From the "hit" definition, and the Poisson assumption, we have:

$$\begin{aligned}
E[H_j|\lambda_j, I_j, T] &= \sum_{k=1}^{\infty} kP[(D_j(T) = k + I_j] \\
&= \sum_{k=1}^{\infty} k \frac{(\lambda_j T)^{k+I_j} e^{-\lambda_j T}}{(k+I_j)!} \\
\text{(set } k + I_j = q) &= \sum_{q=I_j+1}^{\infty} (q - I_j) \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} \\
&= \sum_{q=I_j+1}^{\infty} q \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} - I_j \sum_{q=I_j+1}^{\infty} \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} \\
&= \sum_{q=0}^{\infty} q \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} - \sum_{q=0}^{I_j} q \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} \\
&\quad -I_j \left[\sum_{q=0}^{\infty} \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} - \sum_{q=0}^{I_j} \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} \right] \\
\text{(Exp. val./cum. prob. of Poisson distr.)} &= (\lambda_j T) - \sum_{q=1}^{I_j} \frac{(\lambda_j T)^q e^{-\lambda_j T}}{(q-1)!} - I_j \left[1 - \sum_{q=0}^{I_j} \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} \right] \\
\text{(cpf of Poisson distr)} &= (\lambda_j T) \left[1 - \sum_{q=0}^{I_j - 1} \frac{(\lambda_j T)^q e^{-\lambda_j T}}{q!} \right] - I_j \cdot P[D_j(T) > I_j] \\
\text{(cpf of Poisson distr)} &= (\lambda_j T) \cdot P[D_j(T) > I_j - 1] - I_j \cdot P[D_j(T) > I_j] \qquad (30)
\end{aligned}$$

□

Theorem 3 *For the two-station "hit-covering" problem,*

$$E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2] = P[D_2(T) \le I_2 - 1] - P[D_1(T) \le I_1 - 1] \tag{31}$$

Proof: We have:

$$\begin{aligned}
E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2] &= (E[H_1|\lambda_1, I_1 - 1, T] + E[H_2|\lambda_2, I_2, T]) - (E[H_1|\lambda_1, I_1, T] + E[H_2|\lambda_2, I_2 - 1, T]) \\
\text{(by Lemma 2)} &= (\lambda_1 T) \cdot P[D_1(T) > I_1 - 2] - (I_1 - 1) \cdot P[D_1(T) > I_1 - 1]
\end{aligned}$$

$$
\begin{aligned}
& +(\lambda_2 T)\cdot P[D_2(T) > I_2 - 1] - I_2 \cdot P[D_2(T) > I_2] \\
& -(\lambda_1 T)\cdot P[D_1(T) > I_1 - 1] + I_1 \cdot P[D_1(T) > I_1] \\
& -(\lambda_2 T)\cdot P[D_2(T) > I_2 - 2] + (I_2 - 1)\cdot P[D_2(T) > I_2 - 1] \\
= \; & (\lambda_1 T)\left(P[D_1(T) > I_1 - 2] - P[D_1(T) > I_1 - 1]\right) \\
& -I_1 \left(P[D_1(T) > I_1 - 1] - P[D_1(T) > I_1]\right) + P[D_1(T) > I_1 - 1] \\
& -(\lambda_2 T)\left(P[D_2(T) > I_2 - 2] - P[D_2(T) > I_2 - 1]\right) \\
& +I_2 \left(P[D_2(T) > I_2 - 1] - P[D_2(T) > I_2]\right) - P[D_2(T) > I_2 - 1] \\
= \; & (\lambda_1 T) P[D_1(T) = I_1 - 1] - I_1 P[D_1(T) = I_1] + P[D_1(T) > I_1 - 1] \\
& -(\lambda_2 T) P[D_2(T) = I_2 - 1] + I_2 P[D_2(T) = I_2] - P[D_2(T) > I_2 - 1] \\
\text{(by Lemma 1)} \; = \; & P[D_1(T) > I_1 - 1] - P[D_2(T) > I_2 - 1] \\
= \; & (1 - P[D_1(T) \le I_1 - 1]) - (1 - P[D_2(T) \le I_2 - 1]) \\
= \; & P[D_2(T) \le I_2 - 1] - P[D_1(T) \le I_1 - 1] \qquad (32)
\end{aligned}
$$

□

Theorem 3 provides a convenient test for resolving the two-station "hit covering" problem, under criterion (2):

$$
\begin{aligned}
1 = \arg\min_{j=1,2} E[H_{\mathcal{S}}|j] \quad & \Longleftrightarrow \\
E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2] \le 0 \quad & \Longleftrightarrow \\
P[D_2(T) \le I_2 - 1] - P[D_1(T) \le I_1 - 1] \le 0 \quad & \Longleftrightarrow \\
P[D_1(T) \le I_1 - 1] - P[D_2(T) \le I_2 - 1] \ge 0 \quad & \Longleftrightarrow \\
\frac{P[D_1(T) \le I_1 - 1]}{P[D_2(T) \le I_2 - 1]} \ge 1 \qquad & (33)
\end{aligned}
$$

Equation 33 is the counterpart of Equation 8 for criterion (2), and it will be used in the subsequent analysis, in order to derive the properties of the optimal policy for the "hit covering" problem, when formulated with respect to criterion (2). It is interesting to notice that the difference of the two criteria is expressed analytically by the different right-hand-sides in the last lines of Equations 8 and 33, respectively.

4.1 Case 1: $I_2 \ge I_1$

In this section we show that, similar to the corresponding case for criterion (1), when station S_2 is over-stocked relatively to station S_1, it is optimal to draw the required part from station S_2. To prove this result, we need the following more general property of the Poisson distribution:

Lemma 3 *Let D denote a Poisson-distributed random variable, with rate $\lambda > 0$. Then, for every value $k \in \mathcal{N}$, function*

$$\mathcal{F}(\lambda; k) = P[D \leq k|\lambda]$$

is decreasing *with respect to λ.*

Proof:

$$\begin{aligned} \frac{d}{d\lambda}\mathcal{F}(\lambda; k) &= \frac{d}{d\lambda}\left[e^{-\lambda}\sum_{q=0}^{k}\frac{\lambda^q}{q!}\right] \\ &= -e^{-\lambda}\sum_{q=0}^{k}\frac{\lambda^q}{q!} + e^{-\lambda}\sum_{q=1}^{k}\frac{\lambda^{(q-1)}}{(q-1)!} \\ &= -e^{-\lambda}\frac{\lambda^k}{k!} \; < \; 0 \end{aligned} \tag{34}$$

□

The main result covering this case, is stated and proven as follows:

Theorem 4 *For the two-station "hit covering" problem, formulated according to the criterion of Equation 6, and with*

1. $\lambda_1 \geq \lambda_2$,

2. $I_2 \geq I_1$,

it is always optimal to cover the hit by drawing a part from station S_2.

Proof: We have:

$$\begin{aligned} P[D_2(T) \leq I_2 - 1] &\geq P[D_2(T) \leq I_1 - 1] \quad (\text{since } I_2 \geq I_1) \\ &\geq P[D_1(T) \leq I_1 - 1] \quad (\lambda_1 \geq \lambda_2 \text{ and Lemma 3}) \end{aligned} \tag{35}$$

But then, Equation 35 and 33 together imply that:

$$\arg\min_{j=1,2} E[H_{\mathcal{S}}|j] = 2 \tag{36}$$

□

The following statement is the counterpart of Corollary 1 for criterion (2). It is an immediate consequence of Theorem 4, and the additive structure of Equation 6 with respect to the system stations.

Corollary 3 *In case that $I_i = 1, \ i = 1, \ldots, N$,*

$$i^{*(2)} = \arg\min_j E[H_{\mathcal{S}}|j] = \arg\min_j \lambda_j \tag{37}$$

Finally, Theorems 1 and 4 together imply that, when applied to a two-station "hit covering" problem with $\lambda_1 \geq \lambda_2$ and $I_2 \geq I_1$, the two selection criteria are equivalent. A generalization of this equivalence result for the case of the N-station "hit covering" problem is stated in the following theorem:

Theorem 5 *Consider an N-station "hit covering" problem such that*

$$\exists \ i^* = \arg\min_j \lambda_j \quad s.t. \ \ \forall j \in \{1, \ldots, N\}: \qquad I_j \leq I_{i^*} \tag{38}$$

Then,

$$\arg\max_j P_{NH}(\mathcal{S}|j) = \arg\min_j E[H_{\mathcal{S}}|j] = i^* \tag{39}$$

Proof: The validity of this theorem results immediately from the application of Theorems 1 and 4 to every pair of stations $(S_j, S_{i^*}), \ j \in \{1, \ldots, N\}$. $\square$

4.2 Case 2: $I_1 > I_2$

When $I_1 > I_2$, both stations S_1 and S_2 can be selected as the optimal station for part shipment by application of criterion (2). Similar to the corresponding analysis for criterion (1), the selection outcome for a given two-station problem depends on $\lambda_1, \ \lambda_2, \ I_1, \ I_2$ and T. Furthermore, in this case, it is possible that the selection outcome is different for each of the two criteria. We demonstrate these properties by means of the following examples.

Example 5: Outcome of criterion (1) differs from the outcome of criterion (2) Let us revisit the scenario of Example 2, but now we shall apply criterion (2) for the selection of the covering station. According to Theorem 3, we must consider the difference:

$$P[D_2(T) \leq I_2 - 1] - P[D_1(T) \leq I_1 - 1] = \sum_{k=0}^{0} \frac{2^k e^{-2}}{k!} - \sum_{k=0}^{2} \frac{5^k e^{-5}}{k!} = e^{-2} - e^{-5}(1 + 5 + \frac{25}{2}) = 0.01 \tag{40}$$

Hence, according to Equation 33, the optimal selection is station 2, whereas in Example 2, the selected station was station 1. Indeed, the ratio tests of the last rows in Equations 8 and 33 give:

$$\frac{P[D_1(T) = I_1]}{P[D_2(T) = I_2]} \simeq 0.52 < \frac{P[D_1(T) \leq I_1 - 1]}{P[D_2(T) \leq I_2 - 1]} \simeq 0.921 < 1 \tag{41}$$

□

Example 6: Outcomes of criteria (1) and (2) are the same For $\lambda_1 = 5$, $\lambda_2 = 3$ parts/day, $I_1 = 5$, $I_2 = 1$ parts, and $T = 1$ day:

$$1 < \frac{P[D_1(T) = I_1]}{P[D_2(T) = I_2]} \simeq 1.175 < \frac{P[D_1(T) \leq I_1 - 1]}{P[D_2(T) \leq I_2 - 1]} \simeq 8.8 \tag{42}$$

Hence, according to Equations 8 and 33, station S_1 is selected by both criteria. □

Even though, as indicated by the above examples, the two criteria are *not* equivalent, next, we establish an interesting relationship between them:

Theorem 6 *For the two-station "hit-covering" problem, with $\lambda_1 \geq \lambda_2$, selection of station S_2 by the criterion of Equation 4, implies the selection of S_2 by the criterion of Equation 6, i.e.,*

$$(\lambda_1 \geq \lambda_2) \wedge (i^{*(1)} = \arg\max_{j=1,2} P_{NH}(\mathcal{S}|j) = 2) \Longrightarrow (i^{*(2)} = \arg\min_{j=1,2} E[H_{\mathcal{S}}|j] = 2) \tag{43}$$

Proof: The proof is given in Appendix A. □

An immediate implication of Equation 43, obtained through contraposition, is stated in the following corollary:

Corollary 4 *For the two-station "hit-covering" problem, with $\lambda_1 \geq \lambda_2$,*

$$(i^{*(2)} = \arg\min_{j=1,2} E[H_{\mathcal{S}}|j] = 1) \Longrightarrow (i^{*(1)} = \arg\max_{j=1,2} P_{NH}(\mathcal{S}|j) = 1) \tag{44}$$

An intuitive interpretation of Theorem 6 and Corollary 4 is as follows: For the case where $\lambda_1 \geq \lambda_2$ and $I_1 \leq I_2$, the implications of Equations 43 and 44 result immediately from the fact that both criteria act based on the fact that station S_2 is over-stocked relatively to station S_1. For the complementary case, $\lambda_1 \geq \lambda_2$ and $I_1 > I_2$, each criterion must establish a trade-off for the fact that station S_1 has currently more parts in its inventory that station S_2, but it is also exposed to higher demand. Selecting station S_2 in this situation implies that the decision focuses on the fact that station S_1 must meet higher demand than station S_2, while selection of station S_1 places the emphasis on the inventory sizes. In the light of this remark, Theorem 6 can be interpreted as a tendency of criterion 2 to place more emphasis on the demand element, compared to the emphasis placed on this element by criterion 1. This is further explained by the fact that criterion 1 tries to maximize the probability that "things will turn out right", i.e., the system will receive no hit, ignoring however, how "wrong things can go", once a hit is taken. On the other hand, criterion 2 is rather more considerate of this cumulative effect in its decision, trying to maintain the expected loss at a minimum. In the airline industry context, however, each "hit-covering" decision essentially concerns the period until the next major event – i.e., new part release, or hit occurrence – since every hit is immediately covered. Therefore, we believe that the shorter-term criterion expressed by Equation 4, is more appropriate for this operational context.

We conclude this section by considering the dependency of the optimal selection on variable T. It can be seen from Equation 33 that, similar to the corresponding case of criterion (1), changing the value of variable T can change the final outcome for a two-station "hit covering" problem. However, the dependency of the discriminant function on T is not polynomial anymore, but it involves the exponential $e^{(\lambda_1-\lambda_2)T}$. All the same, we show next that the asymptotic behavior of the optimal policy with respect to criterion (2), for $T \to 0, \infty$, is the same with that of the optimal policy with respect to criterion (1).

Theorem 7 *For the two-station "hit covering" problem,*

1. $\lim_{T\to 0}\{E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2]\} = 0$
2. $\lim_{T\to \infty}\{E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2]\} = 0$

Proof: From Theorem 3,

$$\lim_{T\to 0}\{E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2]\} = \lim_{T\to 0}\{P[D_2(T) \le I_2 - 1] - P[D_1(T) \le I_1 - 1]\} =$$
$$\lim_{T\to 0} P[D_2(T) \le I_2 - 1] - \lim_{T\to 0} P[D_1(T) \le I_1 - 1] = 1 - 1 = 0 \tag{45}$$

where the next to the last equality results from the definition of the Poisson distribution. Also,

$$\lim_{T\to \infty}\{E[H_{\mathcal{S}}|1] - E[H_{\mathcal{S}}|2]\} = \lim_{T\to \infty}\{P[D_2(T) \le I_2 - 1] - P[D_1(T) \le I_1 - 1]\} =$$
$$\lim_{T\to \infty} P[D_2(T) \le I_2 - 1] - \lim_{T\to \infty} P[D_1(T) \le I_1 - 1] = 0 - 0 = 0 \tag{46}$$

where the next to the last equality results by applying L' Hospital's rule [1] on the definition of the Poisson distribution. □

5 The "One-Unit Part Assignment" Problem

The *"one-unit part assignment"* problem can be stated as follows:

The central warehouse has just received a new unit of a certain part, and there are N stations, $S_1, \ldots, S_N$, with pending requests for part replenishment in the system. Specifically, these stations are below the "optimal" safety stock level, estimated by the methodologies presented in [2]. Given that each station $S_j,\ j = 1, \ldots, N$, has an available inventory of I_j part units, and it faces demand D_j which is Poisson-distributed with rate λ_j, assign the received unit to one of these stations so that the system "exposure" to hits over the period T until the next scheduled part release, is minimized. Similar to the "hit covering" problem, we can rationalize the notion of "exposure" by either trying to maximize the probability of taking no hit system-wise, over the period T, or by trying to minimize the expected number of hits, over the same period. The analytical formulations of these optimization problems is *identical* to those expressed by Equations 2 – 6, with the only differing part being the definition of the functions expressing the new inventory statuses $I^{(j)},\ j = 1, \ldots, N$, resulting from the assignment of the part to station $S_j,\ j = 1, \ldots, N$. Specifically, these new functions are defined as follows:

$$I_i^{(j)} = \begin{cases} I_i + 1, & \text{if } i = j \\ I_i, & \text{o.w.} \end{cases} = I_i + \delta_{ij} \tag{47}$$

Notice that since the formulations of the "one-unit part assignment" problem with respect to the two considered criteria have the same form with those of the "hit covering" problem, the decomposability of the optimal decision with respect to the different station pairs is preserved. But when we focus to the problem restriction for $N = 2$, it is easy to see that:

$$\{I_1, I_2\}^{(j)}(\text{1-PA}) = \{I_1 + 1, I_2 + 1\}^{(\bar{j})}(\text{HC}), \quad j = 1, 2 \tag{48}$$

where $\bar{j} = 1 - j$, i.e., it denotes the decision complementary to j. Hence, it follows that the two-station "one-unit part assignment" problem with inventory levels $I_1,\ I_2$ can be *reduced* to a "hit covering" problem with inventory levels $I_1 + 1,\ I_2 + 1$. The solution of the original "one-unit part assignment" problem can be obtained from the solution of the auxiliary "hit covering" problem through *complementation*. Moreover, this reduction allows the extension of the main results obtained for the "hit covering" problem, to the "one-unit part assignment" problem. We state them, without proof, below:

Theorem 8 *For the two-station "one-unit part assignment" problem with*

1. $\lambda_1 \geq \lambda_2$, *and*

2. $I_2 \geq I_1$,

it is always optimal to assign the received part to station S_1, *under both selection criteria.*

Corollary 5 *In case that* $I_i = I,\ \forall i \in \{1, \ldots, N\}$,

$$i^{*(1)}(\textit{1-PA}) = i^{*(2)}(\textit{1-PA}) = \arg\max_j \lambda_j \tag{49}$$

Theorem 9 *For the two-station "one-unit part-assignment" problem, with* $\lambda_1 \geq \lambda_2$,

1.

$$(i^{*(1)}(\textit{1-PA}) = \arg\max_{j=1,2} P_{NH}(\mathcal{S}|j) = 1) \Longrightarrow (i^{*(2)}(\textit{1-PA}) = \arg\min_{j=1,2} E[H_{\mathcal{S}}|j] = 1) \tag{50}$$

2.

$$(i^{*(2)}(\textit{1-PA}) = \arg\min_{j=1,2} E[H_{\mathcal{S}}|j] = 2) \Longrightarrow (i^{*(1)}(\textit{1-PA}) = \arg\max_{j=1,2} P_{NH}(\mathcal{S}|j) = 2) \tag{51}$$

Theorem 10 *As* $T \rightarrow 0, \infty$, *any station* $S_j,\ j = 1, \ldots, N$, *is an optimal selection for the* N*-station "one-unit part assignment" problem, no matter which selection criterion is used, i.e.,*

$$i^{*(1)}(\textit{1-PA}) = i^{*(2)}(\textit{1-PA}) = j, \qquad \forall j \in \{1, \ldots, N\} \tag{52}$$

6 The "n-Unit Part Assignment" Problem

The "n-unit part assignment" problem seeks to distribute a received batch of n part units to N understocked stations, $S_j,\ j = 1, \ldots, N$, each experiencing Poisson-distributed demand, D_j, with rate λ_j. In this section, this problem is addressed for the case that the decision objective is the maximization of the system coverage (i.e., probability of taking no "hit") over a given time horizon T.[1] More specifically, first we show that the optimal allocation of n part units to N stations with *zero* initial inventory can be resolved, under the considered criterion, through a *greedy* algorithm, which allocates one part at a time to these stations, by solving a series of n "one-unit part assignment" problems. Subsequently, we exploit this result in order to develop allocation policies for the more realistic case that the contesting stations have some initial inventory, $I_j,\ j = 1, \ldots, N$.

[1] As it was argued in Section 4.2, this criterion seems to be more appropriate for the airline industry application context (cf. discussion after Corollary 4).

6.1 The "n-unit part assignment" problem with zero initial inventories

This section establishes that the allocation of n part units to N stations with *zero* initial inventory that maximizes the system coverage to demand "hits", can be obtained through a *greedy* algorithm that allocates the n units to these stations, one at a time, through the solution of a sequence of n "one-unit part assignment" problems. Specifically, this algorithm computes the optimal assignment of $k+1$ units to N stations, $S_j,\ j=1,\ldots,N$, by allocating an additional unit to this system of stations, given that their current inventory status, $I_j,\ j=1,\ldots,N$, corresponds to an optimal assignment of k units to these stations.

Let $P^{(j)}_{NH}(I_j)$ denote the probability that station S_j, with demand rate λ_j and inventory level I_j, will experience no "hit" over a certain time horizon, T, i.e.,

$$P^{(j)}_{NH}(I_j) = P[D_j(T) \leq I_j | \lambda_j] \tag{53}$$

The following theorem establishes the optimality of the above greedy algorithm for the two-station case, by stating and proving a slightly stronger result:

Theorem 11 *Let $\{\hat{I}_1, k-\hat{I}_1\}$ be an optimal assignment for a two-station, "k-unit part assignment" problem with zero initial inventories, i.e.,*

$$P^{(1)}_{NH}(\hat{I}_1)P^{(2)}_{NH}(k-\hat{I}_1) \geq P^{(1)}_{NH}(I_1)P^{(2)}_{NH}(k-I_1), \quad \forall I_1 \in \{0,1,\ldots,k\} \tag{54}$$

Then,

1.

$$P^{(1)}_{NH}(\hat{I}_1)P^{(2)}_{NH}(k+1-\hat{I}_1) \geq P^{(1)}_{NH}(I_1)P^{(2)}_{NH}(k+1-I_1), \quad \forall I_1 \in \{0,1,\ldots,k,k+1\};\ I_1 \neq \hat{I}_1, \hat{I}_1+1 \tag{55}$$

2.

$$P^{(1)}_{NH}(\hat{I}_1+1)P^{(2)}_{NH}(k-\hat{I}_1) \geq P^{(1)}_{NH}(I_1)P^{(2)}_{NH}(k+1-I_1), \quad \forall I_1 \in \{0,1,\ldots,k,k+1\};\ I_1 \neq \hat{I}_1, \hat{I}_1+1 \tag{56}$$

Proof: The proof is given in Appendix B. □

The next theorem extends the result of Theorem 11 to the N-station case.

Theorem 12 *Suppose*

$$\{\hat{I}_j,\ j=1,\ldots,N\} = \arg\max_{\{I_j:\sum_{j=1}^{N} I_j=k\}} \prod_{j=1}^{N} P_{NH}^{(j)}(I_j) \tag{57}$$

Then,

$$\exists j^* \in \{1,\ldots,k\}\ s.t.\ \{\tilde{I}_j \equiv \hat{I}_j + \delta_{jj^*};\ j=1,\ldots,N\} = \arg\max_{\{I_j:\sum_{j=1}^{N} I_j=k+1\}} \prod_{j=1}^{N} P_{NH}^{(j)}(I_j) \tag{58}$$

where δ_{jj^*} *denotes* Kronecker's *delta.*

Proof: Pick

$$j^* = \arg\max_{j:\{I_q=\hat{I}_q+\delta_{jq}\}} \prod_{q=1}^{N} P_{NH}^{(q)}(I_q) \tag{59}$$

We shall show, through an *interchange* argument, that the part allocation induced by the definition of j^* in Equation 59, is a maximizer of the last part of Equation 58.

Let us assume that

$$\exists\{I'_j,\ j=1,\ldots,N\} \text{ s.t. } \sum_{j=1}^{N} I'_j = k+1 \text{ and } \prod_{q=1}^{N} P_{NH}^{(q)}(I'_q) > \prod_{q=1}^{N} P_{NH}^{(q)}(\hat{I}_q + \delta_{j^*q}) \tag{60}$$

Clearly, allocation $\{I'_j,\ j=1,\ldots,N\}$ is not one of the allocations implied by the RHS of Equation 59. Therefore, there exists at least one station S_q in it, with $I'_q < \hat{I}_q$. Obviously, there must also exist at least one station S_p with $I'_p > \hat{I}_p$. Focusing on such a pair of stations, $S_q,\ S_p$, we discern two cases:

Case 1: $I'_q + I'_p \geq \hat{I}_q + \hat{I}_p$

Since $\{\hat{I}_j,\ j=1,\ldots,N\}$ is an optimal allocation of k parts to the considered system of N stations, $\{\hat{I}_q, \hat{I}_p\}$ must be an optimal allocation of $\hat{I}_q+\hat{I}_p$ parts to stations S_q and S_p. Furthermore, Theorem 11 implies that, *given an increasing sequence of parts to be shared among two stations, we can always construct a sequence of optimal solutions which will be increasing in* both *solution components.* This observation, combined with the unimodality result of Lemma 7, and the case-defining assumption, imply that

$$P_{NH}^{(q)}(I'_q + |I'_q - \hat{I}_q|) P_{NH}^{(p)}(I'_p - |I'_q - \hat{I}_q|) \geq P_{NH}^{(q)}(I'_q) P_{NH}^{(p)}(I'_p) \tag{61}$$

Case 2: $I'_q + I'_p < \hat{I}_q + \hat{I}_p$

The same line of argument developed for Case 1, now implies that:

$$P_{NH}^{(q)}(I'_q + |I'_p - \hat{I}_p|) P_{NH}^{(p)}(I'_p - |I'_p - \hat{I}_p|) \geq P_{NH}^{(q)}(I'_q) P_{NH}^{(p)}(I'_p) \tag{62}$$

Hence, bringing variable I_q or I_p back to its old level, through this interchange operation, can only improve the overall objective value. Since such an improvement can be incurred every time that a pair of $S_q,\ S_p$ stations, as defined above, is identifiable, it follows that any optimal solution must be dominated by one of the allocations implied by the RHS of Equation 59, which establishes the theorem validity. □

Table 6.1 demonstrates the application of the greedy algorithm in the assignment of five part units to four stations, with demand rates $\lambda_1 = 0.6,\ \lambda_2 = 0.45,\ \lambda_3 = 0.7,\ \lambda_4 = 0.1$, and zero initial inventories. In this example $T = 1$.[2] It can be easily seen that the algorithm complexity is $O(kN)$, where k denotes the number of part units to be allocated and N denotes the number of stations. The alternative approach, based on exhaustive enumeration of all allocation possibilities, would require $\binom{k+N-1}{k}$ evaluations of the objective function.

6.2 The "n-unit part assignment" problem with non-zero initial inventories

The ability to easily identify the optimal allocation of n parts to N stations with *zero* initial inventory, provided by the greedy algorithm developed in the previous section, allows us also to effectively address the "n-unit part assignment" problem for the more practical situation, in which the N stations, $S_j,\ j = 1, \ldots, N$, have some non-zero initial inventory, $I_j,\ j = 1, \ldots, N$. The key observation for developing a solution to this problem is that the optimal assignment of the n new parts should (ideally) lead the system to a status corresponding to the optimal allocation of $n + \sum_{j=1}^{N} I_j$ parts to stations $S_j,\ j = 1, \ldots, N$, starting with zero initial inventories. Hence, the solution proposed to this problem initially uses the greedy algorithm of Section 6.1 to allocate optimally the total of $n+\sum_{j=1}^{N} I_j$ parts to these stations. If it happens that the resulting optimal assignment $\{\hat{I}_j,\ j = 1, \ldots, N\}$ is such that $I_j \leq \hat{I}_j,\ \forall j$, then, each station, S_j, must get $\Delta I_j = \hat{I}_j - I_j$ units from the new lot. Otherwise, – i.e., if $\exists S_j$ with $I_j > \hat{I}_j$ – there are two possibilities: (i) In the case that the allocation of a unit to a station is a *reversible* process, then, we shall allocate the parts of the new lot to the understocked stations, and subsequently we must initiate a sequence of transactions between pairs of understocked and overstocked stations, in order to level out the discrepancy with respect to the optimal allocation. (ii) If, however, allocating a unit to a station is an *irreversible* event, then, the final status of the system is going to be suboptimal w.r.t. the "ideal" allocation, $\{\hat{I}_j,\ j = 1, \ldots, N\}$. In this case, the best strategy is to recognize the stations which are *overstocked* w.r.t. $\{\hat{I}_j,\ j = 1, \ldots, N\}$, and

[2] Alternatively, you can consider the quoted values standing for $\lambda_j T$, instead of λ_j.

	I_1	I_2	I_3	I_4	$P_{NH}(\mathcal{S})$
$k=1$	1	0	0	0	0.2516
	0	1	0	0	0.2280
	0	0	1	0	0.2673 $\Longleftarrow$
	0	0	0	1	0.1730
$k=2$	1	0	1	0	0.4277 $\Longleftarrow$
	0	1	1	0	0.3876
	0	0	2	0	0.3058
	0	0	1	1	0.2940
$k=3$	2	0	1	0	0.4758
	1	1	1	0	0.6201 $\Longleftarrow$
	1	0	2	0	0.4893
	1	0	1	1	0.4705
$k=4$	2	1	1	0	0.6899
	1	2	1	0	0.6634
	1	1	2	0	0.7095 $\Longleftarrow$
	1	1	1	1	0.6822
$k=5$	2	1	2	0	0.7893 $\Longleftarrow$
	1	2	2	0	0.7591
	1	1	3	0	0.7304
	1	1	2	1	0.7805

Table 1: **Example 7:** Allocating 5 parts to 4 stations with $\lambda_1 = 0.6,\ \lambda_2 = 0.45,\ \lambda_3 = 0.7,\ \lambda_4 = 0.1$ and zero initial inventories, through the *greedy* algorithm.

isolate them. Let $\mathcal{S}'$ denote the set of these overstocked stations. Then, the best assignment of the new lot, under the current situation, is obtained by computing the optimal allocation, $\{\bar{I}_j\}$, of $n+\sum_{j=1,\ldots,N:\ S_j \notin \mathcal{S}'} I_j$ parts, to stations $S_j \notin \mathcal{S}'$, and allocating to each of these stations the difference $\Delta I_j = \bar{I}_j - I_j$.

Example 8: As an example for the implementation of this algorithm, consider the allocation of three additional parts to the system in the example of Table 6.1, when the current inventory status is: $I_1 = I_2 = 1$ and $I_3 = I_4 = 0$. From Table 6.1 we see that the ideal allocation of the combined available inventory of five part units is: $\hat{I}_1 = 2,\ \hat{I}_2 = 1,\ \hat{I}_3 = 2,\ \hat{I}_4 = 0$. Therefore, the optimal assignment of the three new parts is: $\Delta I_3 = 2,\ \Delta I_1 = 1$, and $\Delta I_2 = \Delta I_4 = 0$.

If, however, the initial inventory status is: $I_1 = I_4 = 1,\ I_2 = I_3 = 0$, and allocation of a unit to a station is *not* reversible, then, the ideal allocation of Table 6.1 is not implementable; station S_4 is overstocked by one unit. Focusing attention on the remaining part of the system, we find (by applying the greedy algorithm of Section 6.1) that the ideal allocation of the total of four part units to the three stations $S_1,\ S_2,\ S_3$, is: $\bar{I}_1 = 1,\ \bar{I}_2 = 1,\ \bar{I}_3 = 2$. Therefore, the best assignment of the new lot to these stations is: $\Delta I_3 = 2,\ \Delta I_2 = 1,\ \Delta I_1 = 0$. □

Finally, as it was pointed out in the Introduction, in a "real-life" situation, this decision making will be complemented by information and considerations regarding additional issues, like operational costs and transportation constraints. Even in this broader framework, the greedy algorithm of Section 6.1 can provide a quantification of the subpotimality of any existing or tentative allocation of the available parts, relative to the allocation that maximizes the system coverage.

7 Conclusions

In this paper we have developed a formal framework for rationalizing some real-time decisions that must be made in the context of expensive-item, low-volume, distributed inventory systems. More specifically, the problem assumptions were motivated by the operational context of key repairable inventory maintenance in contemporary airline industry. Our results indicate that some of the current practices, developed on an intuitive basis, are correct, but there are also a significant number of cases which require more careful and quantitative analysis. For these cases, the proposed framework offers the required analytical tools, but also assists the further exploration and understanding of the underlying dynamics.

The next step of this research will seek to implement the derived results in the operational environment of an actual airline company. This implementational effort will allow: (i) the validation of the modeling assumptions, (ii) the identification of additional factors that might be crucial in the considered operational context, and the investigation of the potential of their introduction in the decision making framework, and (iii) the assessment of the economic benefits obtained by rationalizing this real-time decision making.

From a more theoretical perspective, we shall investigate how the availability of the greedy algorithm of Section 6.1 can assist the reformulation and solution of the inventory control problems discussed in [2].

Appendix A: Proof of Theorem 6

To simplify the notation, in the subsequent discussion we assume that time is scaled /measured in such a way, that the considered time horizon T is the time unit. Hence $\lambda T = \lambda$. Furthermore, to prove the main theorem result, we need the next two lemmas:

Lemma 4 *Given two positive integers, $I_1, I_2 \in \mathcal{N}^+$, with $I_1 \geq I_2$, the sequence $\frac{I_1+k}{I_2+k}$, $k \in \mathcal{N}$ is decreasing.*

Proof: First, notice that

$$\forall k \in \mathcal{N}, \quad kI_2 \leq kI_1 \iff I_1 I_2 + kI_2 \leq I_1 I_2 + kI_1 \iff \begin{array}{c} I_2 \leq I_1 \iff \\ \dfrac{I_1+k}{I_2+k} \leq \dfrac{I_1}{I_2} \end{array} \tag{63}$$

Then, the main result stated by the lemma is established by means of Equation 63 for $k = 1$, noticing that:

$$\frac{I_1+k+1}{I_2+k+1} = \frac{(I_1+k)+1}{(I_2+k)+1} \leq \frac{I_1+k}{I_2+k} \tag{64}$$

□

Lemma 5 *For the two-station "hit covering" problem with $\lambda_1 \geq \lambda_2$ (> 0), and inventory levels $I_1 > I_2$ (> 0), selection of station S_2 by the criterion of Equation 4 implies that $\frac{\lambda_1}{\lambda_2} \geq \frac{I_1}{I_2}$, i.e.,*

$$(i^{*(1)} = \arg\max_j P_{NH}(\mathcal{S}|j) = 2) \Longrightarrow (\frac{\lambda_1}{\lambda_2} \geq \frac{I_1}{I_2}) \tag{65}$$

Proof: The hypothesis of Equation 65, together with Equations 8 and 9, imply that:

$$\left[\sum_{k=0}^{I_1-1}\frac{\lambda_1^k}{k!}\right]\frac{\lambda_2^{I_2}}{I_2!}\leq\left[\sum_{k=0}^{I_2-1}\frac{\lambda_2^k}{k!}\right]\frac{\lambda_1^{I_1}}{I_1!}\quad\Longrightarrow$$
$$\left[\sum_{k=0}^{I_1-1}\frac{I_1!}{k!}\lambda_1^{k-I_1}\right]\leq\left[\sum_{k=0}^{I_2-1}\frac{I_2!}{k!}\lambda_2^{k-I_2}\right]\quad\Longrightarrow$$
$$\left[\sum_{k=I_1-I_2}^{I_1-1}\frac{I_1!}{k!}\lambda_1^{k-I_1}\right]\leq\left[\sum_{k=0}^{I_2-1}\frac{I_2!}{k!}\lambda_2^{k-I_2}\right]\quad\Longrightarrow$$
$$\left[\sum_{k=0}^{I_2-1}\frac{I_1!}{(k+I_1-I_2)!}\lambda_1^{k-I_2}\right]\leq\left[\sum_{k=0}^{I_2-1}\frac{I_2!}{k!}\lambda_2^{k-I_2}\right]\tag{66}$$

Next, suppose that the conclusion of Equation 65 does not hold true, i.e.,

$$\frac{\lambda_1}{\lambda_2}<\frac{I_1}{I_2}\tag{67}$$

Then, Equation 67 and Lemma 4 imply that:

$$\forall k\in\{0,\ldots,I_2-1\},\ \forall q\in\{1,\ldots,I_2-k\},\quad\frac{k+q+I_1-I_2}{k+q}>\frac{\lambda_1}{\lambda_2}\tag{68}$$

and therefore,

$$\forall k\in\{0,\ldots,I_2-1\},\quad\prod_{q=1}^{I_2-k}\frac{k+q+I_1-I_2}{k+q}>\left(\frac{\lambda_1}{\lambda_2}\right)^{I_2-k}\quad\Longleftrightarrow$$
$$\frac{I_1!}{(k+I_1-I_2)!}\lambda_1^{k-I_2}>\frac{I_2!}{k!}\lambda_2^{k-I_2}\tag{69}$$

Adding Equation 69 for all $k\in\{0,\ldots,I_2-1\}$, gives:

$$\left[\sum_{k=0}^{I_2-1}\frac{I_1!}{(k+I_1-I_2)!}\lambda_1^{k-I_2}\right]>\left[\sum_{k=0}^{I_2-1}\frac{I_2!}{k!}\lambda_2^{k-I_2}\right]\tag{70}$$

which contradicts the last row of Equation 66, and establishes the result. □

Now we are ready to prove the main result of Theorem 6. We distinguish two main cases:

Case 1: $I_2\geq I_1$

In this case, the implication of Theorem 6 is immediately established by the Theorems 1 and 4.

Case 2: $I_1>I_2$

Then, the hypothesis of Theorem 6, together with Equations 8 and 9, imply that:

$$\left[\sum_{k=0}^{I_1-1}\frac{\lambda_1^k}{k!}\right]\frac{\lambda_2^{I_2}}{I_2!}\leq\left[\sum_{k=0}^{I_2-1}\frac{\lambda_2^k}{k!}\right]\frac{\lambda_1^{I_1}}{I_1!}\tag{71}$$

Furthermore, Lemmas 4 and 5 together imply that:

$$\forall k \in \mathcal{N}, \quad \frac{I_1 + k}{I_2 + k} \leq \frac{\lambda_1}{\lambda_2} \iff$$
$$\frac{\lambda_2}{I_2 + k} \leq \frac{\lambda_1}{I_1 + k} \tag{72}$$

Equations 71 and 72 together imply that:

$$\forall q \in \mathcal{N}, \quad \left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] \frac{\lambda_2^{I_2+q}}{(I_2+q)!} \leq \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] \frac{\lambda_1^{I_1+q}}{(I_1+q)!} \tag{73}$$

Taking the sum of Equations 73 for all q, we get:

$$\left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] \left[\sum_{k=I_2}^{\infty} \frac{\lambda_2^k}{k!}\right] \leq \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] \left[\sum_{k=I_1}^{\infty} \frac{\lambda_1^k}{k!}\right] \iff$$
$$\left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] + \left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] \left[\sum_{k=I_2}^{\infty} \frac{\lambda_2^k}{k!}\right] \leq \left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] \left[\sum_{k=I_2}^{I_2-1} \frac{\lambda_2^k}{k!}\right] + \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] \left[\sum_{k=I_1}^{\infty} \frac{\lambda_1^k}{k!}\right] \iff$$
$$\left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] \left[\sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!}\right] \leq \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] \left[\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!}\right] \iff$$
$$\left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] e^{\lambda_2} \leq \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] e^{\lambda_1} \iff$$
$$\left[\sum_{k=0}^{I_1-1} \frac{\lambda_1^k}{k!}\right] e^{-\lambda_1} \leq \left[\sum_{k=0}^{I_2-1} \frac{\lambda_2^k}{k!}\right] e^{-\lambda_2} \iff$$
$$P[D_1 \leq I_1 - 1] \leq P[D_2 \leq I_2 - 1] \iff$$
$$i^{*(2)} = \arg\min_{j=1,2} E[H_{\mathcal{S}}|j] = 2 \tag{74}$$

Since Cases 1 and 2 cover exhaustively all the relative inventory statuses arising in a two-station system, the result of Theorem 6 has been established. □

Appendix B: Proof of Theorem 11

In the following, let $HC(\lambda_1, \lambda_2, I_1, I_2)$ denote the optimal selection for a two-station "hit covering" problem with demand rates λ_1, λ_2, and inventory levels I_1, I_2, with respect to the criterion of Equation 4. To prove Theorem 11, we need the following two lemmas:

Lemma 6

$$HC(\lambda_1, \lambda_2, I_1, I_2) = S_1 \Longrightarrow HC(\lambda_1, \lambda_2, I_1 + 1, I_2) = S_1 \tag{75}$$

Proof: (By contradiction) Let the optimal selection for the new "hit covering" problem be station S_2. Then, from Equation 8, we get:

$$\begin{aligned}
P[D_1 \le I_1]P[D_2 = I_2] < P[D_2 \le I_2 - 1]P[D_1 = I_1 + 1] &\implies \\
\left[\sum_{q=0}^{I_1} \frac{\lambda_1^q}{q!} e^{-\lambda_1}\right] \frac{\lambda_2^{I_2}}{I_2!} e^{-\lambda_2} < \left[\sum_{q=0}^{I_2-1} \frac{\lambda_2^q}{q!} e^{-\lambda_2}\right] \frac{\lambda_1^{I_1+1}}{(I_1+1)!} e^{-\lambda_1} &\implies \\
\frac{I_1+1}{\lambda_1}\frac{\lambda_2^{I_2}}{I_2!} + \left[\sum_{q=1}^{I_1} \frac{\lambda_1^{q-1}}{q!}(I_1+1)\right] \frac{\lambda_2^{I_2}}{I_2!} < \left[\sum_{q=0}^{I_2-1} \frac{\lambda_2^q}{q!}\right] \frac{\lambda_1^{I_1}}{I_1!} &\implies \\
\left[\sum_{q=0}^{I_1-1} \frac{\lambda_1^q}{q!}\frac{I_1+1}{q+1}\right] \frac{\lambda_2^{I_2}}{I_2!} < \left[\sum_{q=0}^{I_2-1} \frac{\lambda_2^q}{q!}\right] \frac{\lambda_1^{I_1}}{I_1!} &\implies \\
\left[\sum_{q=0}^{I_1-1} \frac{\lambda_1^q}{q!}\right] \frac{\lambda_2^{I_2}}{I_2!} < \left[\sum_{q=0}^{I_2-1} \frac{\lambda_2^q}{q!}\right] \frac{\lambda_1^{I_1}}{I_1!} &\implies \\
P[D_1 \le I_1 - 1]P[D_2 = I_2] < P[D_2 \le I_2 - 1]P[D_1 = I_1] &
\end{aligned} \tag{76}$$

But the last line of Equation 76 implies that the optimal selection for the two-station "hit covering" problem with demand rates λ_1, λ_2 and inventory levels I_1, I_2 is station S_2 (cf. Eq. 8), which contradicts the initial assumption. □

Lemma 7 *Function* $\mathcal{G}(I) := P_{NH}^{(1)}(I)P_{NH}^{(2)}(k-I)$, $I \in \{0, 1, \ldots, k\}$, *is* unimodal *for every* k.

Proof: For any $I \in \{0, 1, \ldots, k\}$, we have:

$$\begin{aligned}
\mathcal{G}(I+1) - \mathcal{G}(I) &= \\
P_{NH}^{(1)}(I+1)P_{NH}^{(2)}(k-I-1) - P_{NH}^{(1)}(I)P_{NH}^{(2)}(k-I) &= \\
\left[\sum_{q=0}^{I+1} \frac{\lambda_1^q}{q!} e^{-\lambda_1}\right]\left[\sum_{q=0}^{k-I-1} \frac{\lambda_2^q}{q!} e^{-\lambda_2}\right] - \left[\sum_{q=0}^{I} \frac{\lambda_1^q}{q!} e^{-\lambda_1}\right]\left[\sum_{q=0}^{k-I} \frac{\lambda_2^q}{q!} e^{-\lambda_2}\right] &= \\
\left[\sum_{q=0}^{k-I-1} \frac{\lambda_2^q}{q!} e^{-\lambda_2}\right] \frac{\lambda_1^{I+1}}{(I+1)!} e^{-\lambda_1} - \left[\sum_{q=0}^{I} \frac{\lambda_1^q}{q!} e^{-\lambda_1}\right] \frac{\lambda_2^{k-I}}{(k-I)!} e^{-\lambda_2} &= \\
P[D_2 \le k - I - 1]P[D_1 = I + 1] - P[D_1 \le I]P[D_2 = k - I] &
\end{aligned} \tag{77}$$

Then, Equations 8, 77, and Lemma 6 imply:

$$\begin{aligned}
\mathcal{G}(I+1) - \mathcal{G}(I) > 0 &\implies \\
HC(\lambda_1, \lambda_2, I+1, k-I) = S_2 &\implies \\
HC(\lambda_1, \lambda_2, I+1, k-I+1) = S_2 &\implies \\
HC(\lambda_1, \lambda_2, I, k-I+1) = S_2 &\implies \\
\mathcal{G}(I) - \mathcal{G}(I-1) > 0 &
\end{aligned} \tag{78}$$

Similarly,

$$\begin{aligned}
\mathcal{G}(I+1) - \mathcal{G}(I) < 0 &\implies \\
HC(\lambda_1, \lambda_2, I+1, k-I) = S_1 &\implies \\
HC(\lambda_1, \lambda_2, I+2, k-I) = S_1 &\implies \\
HC(\lambda_1, \lambda_2, I+2, k-I-1) = S_1 &\implies \\
\mathcal{G}(I+2) - \mathcal{G}(I+1) < 0 & \qquad (79)
\end{aligned}$$

Equations 78 and 79 together imply that function $\mathcal{G}(I)$, $I \in \{0, 1, \ldots, k\}$, can have only one local maximum, and therefore, it is unimodal. □

Now we are ready to prove the validity of Theorem 11. First, we prove the validity of Equation 55. W.l.o.g., let us assume that $\lambda_1 \leq \lambda_2$ (o.w., switch stations S_1 and S_2). The unimodality of function $\mathcal{G}(I_1; k+1)$, established by Lemma 7, implies that it is sufficient to show the validity of Equation 55 for $I_1 = \hat{I}_1 - 1,\ \hat{I}_1 + 2$.

(i) $I_1 = \hat{I}_1 - 1$

We have:

$$\begin{aligned}
P_{NH}^{(1)}(\hat{I}_1) P_{NH}^{(2)}(k+1-\hat{I}_1) - P_{NH}^{(1)}(\hat{I}_1 - 1) P_{NH}^{(2)}(k+2-\hat{I}_1) &= \\
(P[D_1 \leq \hat{I}_1 - 1] + P[D_1 = \hat{I}_1]) P[D_2 \leq k+1-\hat{I}_1] & \\
-P[D_1 \leq \hat{I}_1 - 1](P[D_2 \leq k+1-\hat{I}_1] + P[D_2 = k+2-\hat{I}_1]) &= \\
P[D_2 \leq k+1-\hat{I}_1] P[D_1 = \hat{I}_1] - P[D_1 \leq \hat{I}_1 - 1] P[D_2 = k+2-\hat{I}_1] & \qquad (80)
\end{aligned}$$

Furthermore, Equation 54 implies that:

$$HC(\lambda_1, \lambda_2, \hat{I}_1, k+1-\hat{I}_1) = S_2 \qquad (81)$$

and therefore, by Lemma 6,

$$HC(\lambda_1, \lambda_2, \hat{I}_1, k+2-\hat{I}_1) = S_2 \qquad (82)$$

But then, Equations 82, 8, and 80 imply that:

$$P_{NH}^{(1)}(\hat{I}_1) P_{NH}^{(2)}(k+1-\hat{I}_1) \geq P_{NH}^{(1)}(\hat{I}_1 - 1) P_{NH}^{(2)}(k+2-\hat{I}_1) \qquad (83)$$

(ii) $I_1 = \hat{I}_1 + 2$

We have:

$$P_{NH}^{(1)}(\hat{I}_1)P_{NH}^{(2)}(k+1-\hat{I}_1) \geq P_{NH}^{(1)}(\hat{I}_1+2)P_{NH}^{(2)}(k-\hat{I}_1-1) \iff$$
$$P[D_1 \leq \hat{I}_1](P[D_2 \leq k-\hat{I}_1-1] + P[D_2 = k-\hat{I}_1] + P[D_2 = k-\hat{I}_1+1]) \geq$$
$$(P[D_1 \leq \hat{I}_1] + P[D_1 = \hat{I}_1+1] + P[D_1 = \hat{I}_1+2])P[D_2 \leq k-\hat{I}_1-1] \iff$$
$$P[D_1 \leq \hat{I}_1]P[D_2 = k-\hat{I}_1] - P[D_2 \leq k-\hat{I}_1-1]P[D_1 = \hat{I}_1+1] \geq$$
$$P[D_2 \leq k-\hat{I}_1-1]P[D_1 = \hat{I}_1+2] - P[D_1 \leq \hat{I}_1]P[D_2 = k+1-\hat{I}_1] \tag{84}$$

Equation 54 implies that

$$HC(\lambda_1, \lambda_2, \hat{I}_1+1, k-\hat{I}_1) = S_1 \tag{85}$$

which, by means of Equation 8, implies that

$$P[D_1 \leq \hat{I}_1]P[D_2 = k-\hat{I}_1] - P[D_2 \leq k-\hat{I}_1-1]P[D_1 = \hat{I}_1+1] \geq 0 \tag{86}$$

But then, Equations 84 and 86 together imply that in order to establish the validity of Equation 55 for $I_1 = \hat{I}_1 + 2$, it suffices to show that

$$P[D_2 \leq k-\hat{I}_1-1]P[D_1 = \hat{I}_1+2] - P[D_1 \leq \hat{I}_1]P[D_2 = k+1-\hat{I}_1] \leq 0 \iff$$
$$P[D_2 \leq k-\hat{I}_1-1]P[D_1 = \hat{I}_1+1]\frac{\lambda_1}{\hat{I}_1+2} - P[D_1 \leq \hat{I}_1]P[D_2 = k-\hat{I}_1]\frac{\lambda_2}{k+1-\hat{I}_1} \leq 0 \tag{87}$$

Furthermore, Equation 86 implies that for Equation 87 to hold true, it is sufficient that

$$\frac{\lambda_1}{\hat{I}_1+2} \leq \frac{\lambda_2}{k+1-\hat{I}_1} \iff \frac{\lambda_2}{\lambda_1} \geq \frac{k+1-\hat{I}_1}{\hat{I}_1+2} \tag{88}$$

Equation 88 is proven as follows:

(a) $k - \hat{I}_1 \leq \hat{I}_1 + 1$

Then,

$$k-\hat{I}_1 \leq \hat{I}_1+1 \iff k+1-\hat{I}_1 \leq \hat{I}_1+2 \iff \frac{k+1-\hat{I}_1}{\hat{I}_1+2} \leq 1 \leq \frac{\lambda_2}{\lambda_1} \tag{89}$$

where the last inequality holds from the assumption $\lambda_1 \leq \lambda_2$.

(b) $k - \hat{I}_1 > \hat{I}_1 + 1$

Then, Equation 85, together with the assumption $\lambda_1 \leq \lambda_2$, and Lemma 5, imply that

$$\frac{\lambda_2}{\lambda_1} \geq \frac{k-\hat{I}_1}{\hat{I}_1+1} \tag{90}$$

and, by means of Lemma 4,

$$\frac{\lambda_2}{\lambda_1} \geq \frac{k-\hat{I}_1}{\hat{I}_1+1} \geq \frac{k+1-\hat{I}_1}{\hat{I}_1+2} \tag{91}$$

This completes the proof of the validity of Equation 55. The validity of Equation 56 follows immediately from the validity of Equation 55 through the symmetry (*duality*) that exists between stations S_1 and S_2. More specifically, the equivalent of Equation 55 for station S_2 is

$$P_{NH}^{(2)}(\hat{I}_2)P_{NH}^{(1)}(k+1-\hat{I}_2) \geq P_{NH}^{(2)}(I_2)P_{NH}^{(1)}(k+1-I_2), \quad \forall I_2 \in \{0, 1, \ldots, k, k+1\};\ I_2 \neq \hat{I}_2, \hat{I}_2+1 \quad (92)$$

Substituting $\hat{I}_2 = k - \hat{I}_1$ and $I_2 = k + 1 - I_1$ in Equation 92, we get Equation 56. □

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