Problem 8.11

Our model is a Jackson network of five nodes. Let's calculate the actual arrival rates \( \lambda_i \) at each node \( i \):

\[
\begin{align*}
\lambda_1 &= \lambda_1 + \lambda_2 \times 0.4, \\
\lambda_2 &= \lambda_2 + \lambda_3 \times 0.8, \\
\lambda_3 &= \lambda_2 + \lambda_3, \\
\lambda_4 &= \lambda_4 + \lambda_3 \times 0.3, \\
\lambda_5 &= \lambda_5 + \lambda_3 \times 0.9
\end{align*}
\]

\[
\begin{align*}
\lambda_1 &= \frac{5}{3}, \\
\lambda_2 &= \frac{5}{3}, \\
\lambda_3 &= \frac{5}{3}, \\
\lambda_4 &= \frac{5}{3}, \\
\lambda_5 &= \frac{5}{3}
\end{align*}
\]

\[ \min^{-1} \]

a) Each node \( H_i \) can be treated as an \( M/M/1 \) queueing system with arrival rate \( \lambda_i \) and service rate \( \mu_i \).

Hence,

\[
\begin{align*}
\rho_1 &= \frac{\lambda_1}{\mu_1} = \frac{5/3}{5} = \frac{5}{6}, \\
\rho_2 &= \frac{\lambda_2}{\mu_2} = \frac{5/3}{5} = \frac{5}{9}, \\
\rho_3 &= \frac{\lambda_3}{\mu_3} = \frac{5/3}{5} = \frac{5}{9}, \\
\rho_4 &= \frac{\lambda_4}{\mu_4} = \frac{5/3}{5} = \frac{5}{9}, \\
\rho_5 &= \frac{\lambda_5}{\mu_5} = \frac{5/3}{5} = \frac{5}{9}
\end{align*}
\]

b) The throughput of the system is \( \frac{\lambda_3 \rho_3}{\sum \rho_i} \). This should be expected since all nodes are stable (all \( \rho_i < 1 \)) and the total incoming rate is \( \lambda_1 = 1 \min^{-1} \).

c) If \( X_i \) is the queue length at node \( H_i \), i=1,2,...,5 then the average total number of customers in the system, \( E[X] \), is

\[
E[X] = \sum_{i=1}^{5} E[X_i] = \sum_{i=1}^{5} \frac{\rho_i}{1 - \rho_i} = \frac{5/6}{1-5/6} + \frac{5/9}{1-5/9} + \frac{5/9}{1-5/9} + \frac{5/9}{1-5/9} + \frac{5/9}{1-5/9}
\]

\[
= 5 + 9.0 + \frac{50}{8} + 2 + 2 = 39.125
\]

The arrival rate \( \lambda \) into the system is 

\[ \lambda = \lambda_1 = 1 \]

By Little's law, the average system time \( E[S] \) is

\[
E[S] = \frac{E[X]}{\lambda} = \frac{39.125}{1} = 39.125
\]

110
d. \[ P(X_2 > 3) = 1 - P(X_2 = 0) - P(X_2 = 1) - P(X_2 = 2) - P(X_2 = 3) \]

\[ = 1 - (1 - p_2) - (1 - p_2) p_2 - (1 - p_2) p_2^2 - (1 - p_2) p_2^3 \]

\[ = 1 - \frac{1}{21} - \frac{1}{21} \frac{20}{21} - \frac{1}{21} \left( \frac{20}{21} \right)^2 - \frac{1}{21} \left( \frac{20}{21} \right)^3 \]

\[ = 0.893 \]
Problem 8.14

Following the notation that we used in the discussion of the mean value analysis for closed QNs, let

\[ W_m(j) = \text{average response time at node } j \text{ for an } m \text{-customer network.} \]

Then, for part (a), we essentially need to compute the quantity

\[ W_2(2) + W_2(3) \]

In the provided figure, \( j = 1, 2, 3 \), and a "ring" topology for the nodes.

This last topology implies the following routing matrix \( P \): 

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

and therefore the quantities \( T_{ij} \) introduced in the discussion on QNs are computed as follows:

\[
\begin{cases}
(T_1, T_2, T_3) = (T_1, T_2, T_3) \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \\
\sum_{j=1}^{3} T_{ij} = 1
\end{cases}
\]

\[
\Rightarrow T_1 = T_2 = T_3 = \frac{1}{3}
\]

Then, applying the recursion:

\[
\begin{cases}
W_m(j) = \frac{1}{K_j} + \frac{(m-1) T_{ij} W_{m-1}(j)}{K_j} - \sum_{i=1}^{3} T_{ij} W_{m-1}(i) \quad \forall j \\
W_1(j) = \frac{1}{K_j}, \forall j
\end{cases}
\]
But as derived during the discussion of MVA, we obtain:

\[
W_2(2) = \frac{1}{\nu_2} + \frac{(2-1) \nu_3 W_1(2)}{\nu_2 \left( \nu_3 W_1(1) + \nu_3 W_1(2) + \nu_3 W_1(3) \right)} = \frac{1}{\nu_2} \left[ 1 + \frac{\nu_3}{\nu_3 + \nu_3 + \nu_3} \right] = \frac{1}{\nu_2} \left[ 1 + \frac{\nu_3}{\nu_3 + \nu_3 + \nu_3} \right] = 1.4 \text{ min}
\]

\[
W_2(3) = \frac{1}{\nu_3} + \frac{(2-1) \nu_3 W_1(3)}{\nu_2 \left( \nu_3 W_1(1) + \nu_3 W_1(2) + \nu_3 W_1(3) \right)} = \frac{1}{\nu_3} \left[ 1 + \frac{\nu_3}{\nu_3 + \nu_3 + \nu_3} \right] = \frac{1}{\nu_3} \left[ 1 + \frac{\nu_3}{\nu_3 + \nu_3 + \nu_3} \right] = 1.11 \text{ min}
\]

and the sought expected time is

\[
W_2(2) + W_2(3) \approx 1.4 + 1.11 = 2.51 \text{ min}.
\]

A less mechanistic way to derive the above results in the considered case is as follows:

Let also:

- \( E \left( m \right) \) denote the expected number of customers at station \( j \) for an \( m \)-customer network

- \( T \left( m \right) \) denote the throughput for an \( m \)-customer network

- \( T (m) \) denote the total expected time for going through all

Then, from Little's law (applied to the entire network):
\[ \overline{TH(1)} = \frac{1}{C(1,1)} = \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} \]

Also, application of Little's law at nodes 2 and 3 gives:

\[ L_2(2) = \overline{TH(1)} \cdot \frac{1}{\mu_2} \quad \text{and} \quad L_2(3) = \overline{TH(1)} \cdot \frac{1}{\mu_3} \]

and from the arrival theorem:

\[ W_2(2) = \frac{1}{\mu_2} \left( 1 + L_2(2) \right) \]
\[ W_2(3) = \frac{1}{\mu_3} \left( 1 + L_2(3) \right) \]

which gives:

\[ W_2(2) = \frac{1}{\mu_2} \left[ 1 + \frac{\frac{1}{\mu_2}}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} \right] \]
\[ W_2(3) = \frac{1}{\mu_3} \left[ 1 + \frac{\frac{1}{\mu_3}}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} \right] \]

as before.

For part (b), just compute also

\[ W_2(1) = \frac{1}{\mu_1} \left[ 1 + \frac{\frac{1}{\mu_2}}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} \right] = \frac{1}{0.5} \left[ 1 + \frac{1.5}{1.5 + \frac{1}{1} + \frac{1}{1.2}} \right] \approx 0.94 \text{ min} \]

Then

\[ C(T(2)) = W_2(1) + W_2(2) + W_2(3) \approx 0.94 + 0.94 + 0.51 \approx 2.51 \text{ min} \]

and

\[ \overline{TH(2)} = \frac{2}{C(T(2))} = \frac{2}{3.35} \approx 0.597 \text{ min}^{-1} \approx 0.6 \text{ min}^{-1} \]
Problem 8.15:

a)

b) Let's first calculate the stationary state probabilities:

\[ \begin{align*}
    \mu_1 \pi(200) &= \mu_2 \pi(101) \\
    (\mu_2 + \mu_3) \pi(110) &= \mu_2 \pi(110) + \mu_3 \pi(011) \\
    \mu_2 \pi(110) &= \mu_1 \pi(110) \\
    (\mu_2 + \mu_3) \pi(011) &= \mu_2 \pi(011) + \mu_3 \pi(011) \\
    \mu_3 \pi(011) &= \mu_1 \pi(011) \\
    \mu_3 \pi(011) &= \mu_2 \pi(011) + \mu_3 \pi(011) \\
\end{align*} \]

\[ \Rightarrow \begin{align*}
    1.5 \pi(200) &= 1.2 \pi(101) \\
    2.5 \pi(110) &= 1.5 \pi(200) + 1.2 \pi(011) \\
    \pi(110) &= 1.5 \pi(110) \\
    2.2 \pi(011) &= \pi(110) + 1.5 \pi(101) \\
    1.7 \pi(011) &= \pi(011) \\
    2.7 \pi(101) &= \pi(110) + 1.2 \pi(011) \\
\end{align*} \]

We also have that

\[ \pi(200) + \pi(110) + \pi(110) + \pi(011) + \pi(011) + \pi(011) = 1 \]

Solving the above system of linear equations we get:

\[ \begin{align*}
    \pi(200) &= \frac{598}{5587} \\
    \pi(101) &= \frac{740}{5587} \\
    \pi(110) &= \frac{888}{5587} \\
    \pi(011) &= \frac{1110}{5587} \\
    \pi(011) &= \frac{925}{5587} \\
    \pi(011) &= \frac{1359}{5587} \\
\end{align*} \]

\[ \text{119} \]
P\{any one node in the system is blocked\} = \pi(011) + \pi(101) = \frac{0.925}{5584} + \frac{1.332}{5584} = \frac{2.257}{5584} \approx 0.407

\[\pi(011) = \pi(010) + \pi(101) = \frac{569}{5584} + \frac{224}{5584} = \frac{793}{5584} \approx 0.138\]

It is also interesting to compute the throughput attained by this new configuration and compare it to the configuration throughput attained by the configuration of problem 8.14.

This new throughput is given by:

\[TH' = \mu_2 \left[ \pi(011) + \pi(111) + \pi(101) \right] = \frac{740}{5584} + \frac{150}{5584} + \frac{985}{5584} \approx 0.536 \text{ min}^{-1}\]

In this case, the blocking taking place in this new configuration does not have a substantial impact on the system throughput.

But, in general, the reduction of throughput due to introduced blocking effects can be significant.

* For an explanation of the throughput equality observed between problems 8.14 and 8.15 see the very pertinent remarks provided in the next 3 pages; these pages are the solution of problem 8.15 by one of the students of this course.
\[ L_1(1) = \lambda_1 \cdot W_1(1) = \lambda_1 \cdot \frac{1}{1.5} \]
\[ L_2(1) = \lambda_2 \cdot W_2(1) = \lambda_2 \cdot \frac{1}{1.2} \]
\[ L_3(1) + L_2(1) + L_1(1) = \lambda_1 \left( \frac{1}{1.5} + 1 + \frac{1}{1.2} \right) = 1 \]
\[ \Rightarrow \lambda_1 = 0.4 \]
\[ \Rightarrow L_1(1) = 0.2667, \quad L_2(1) = 0.4, \quad L_3(1) = 0.3333 \]

For \( N=2 \).
\[ W_1(2) = \frac{1}{\mu_1} + \frac{1}{\mu_1} \cdot L_1(1) = 0.8445 \]
\[ W_2(2) = \frac{1}{\mu_2} + \frac{1}{\mu_2} \cdot L_2(1) = 1.4 \]
\[ W_3(2) = \frac{1}{\mu_3} + \frac{1}{\mu_3} \cdot L_3(1) = 1.1111 \]
\[ L_1(2) = \lambda_1 \cdot W_1(2) = \lambda_1 \cdot 0.8445 \]
\[ L_2(2) = \lambda_2 \cdot W_2(2) = \lambda_2 \cdot 1.4 \]
\[ L_3(2) = \lambda_3 \cdot W_3(2) = \lambda_3 \cdot 1.1111 \]
\[ \Rightarrow L_1(2) + L_2(2) + L_3(2) = \lambda_1 (0.8445 + 1.4 + 1.1111) = 2 \]
\[ \lambda_2 (0.8445 + 1.4 + 1.1111) = 2 \]
\[ \Rightarrow \lambda_2 = 0.596 \]
\[ (a) \quad W_1(2) + W_2(2) = 1.4 + 1.1111 = 2.5111 \]
\[ (b) \quad \lambda_2 = 0.596 \]

8.15 in text book

(a)
In state space, number with prime indicates job blocked by subsequent node.

b) Following flow balance equations should be satisfied in the equilibrium.

for state (2,0,0) \( \mu_1 \cdot \pi(2,0,0) = \mu_3 \cdot \pi(1,0,1) \)

for state (1,1,0) \( \mu_1 \cdot \pi(1,1,0) + \mu_2 \cdot \pi(1,1,0) = \mu_1 \cdot \pi(2,0,0) + \mu_3 \cdot \pi(0,1,1) \)

for state (1',1,0) \( \mu_2 \cdot \pi(1',1,0) = \mu_1 \cdot \pi(1,1,0) \)

for state (0,1,1) \( \mu_2 \cdot \pi(0,1,1) = \mu_2 \cdot \pi(1',1,0) + \mu_1 \cdot \pi(1,0,1) \)

for state (1,0,1) \( \mu_1 \cdot \pi(1,0,1) = \mu_2 \cdot \pi(1,1,0) + \mu_3 \cdot \pi(0,1,1) \)

for state (0,1',1) \( \mu_3 \cdot \pi(0,1',1) = \mu_2 \cdot \pi(0,1,1) \)

by distribution property \( \sum_{\pi(s)} \pi(s) = 1. \)

If we solve the above equations, we can get stationary probability for each state.

I got the following result by several matrix operation using Matlab.

\[
\begin{bmatrix}
\pi(2,0,0) \\
\pi(1,1,0) \\
\pi(1',1,0) \\
\pi(0,1,1) \\
\pi(1,0,1) \\
\pi(0,1',1)
\end{bmatrix} = \begin{bmatrix}
0.1060 \\
0.1589 \\
0.2383 \\
0.1987 \\
0.1325 \\
0.1656
\end{bmatrix}
\]

→ probability that any customers are blocked:

\[\pi(1',1,0) + \pi(0,1',1) = 0.2383 + 0.1656 = 0.4039\]

Further, I think alternative way using special structure of this problem.

What if we replace (1',1,0) (respectively (0,1',1)) with (0,2,0) (respectively (0,0,2))?

Since there is only two customers in the system, the state diagram of (a) will not change at all. In other words, we can substitute (0,2,0) and (0,0,2) for (1',1,0) and (0,1',1) respectively without any change of state diagram. Moreover, the changed state diagram is nothing more than state diagram of network without any capacity constraints.

Therefore, we can analyze this system as the normal closed network, while considering state (0,2,0) and (0,0,2) as blocking situation (1',1,0) and (0,1',1) respectively.

If this is normal closed network, we can assume that stationary distribution is product form as follows. Since relative arrival rates are same, we can assume that \( \lambda = 1. \)

→ \( \rho_1 = \frac{1}{1.5} = 0.667, \quad \rho_2 = 1, \quad \rho_3 = 0.833 \)

→ \( \pi(n_1,n_2,n_3) = A \cdot 0.667^{n_1} \cdot 0.833^{n_3} \)
Let's get a normalization constant $A$.

$$
\sum_{(n_1, n_2, n_3)} \pi(n_1, n_2, n_3) = A \cdot \sum_{(n_1, n_2, n_3)} 0.667^{n_1} \cdot 0.833^{n_3} = 1 \n$$

$$
\rightarrow A(0.667^2 \cdot 0.833^0 + 0.667^0 \cdot 0.833^0 + 0.667^0 \cdot 0.833^2 + 0.667^1 \cdot 0.833^0 + 0.667^1 \cdot 0.833^1 + 0.667^0 \cdot 0.833^1) = 1 \n$$

$$
\rightarrow A = 0.2384 \n$$

In this new state diagram, (0.2, 0) and (0.0, 2) indicate blocking situation.

$$
\pi(0.2, 0) + \pi(0, 0.2) = 0.2384 \cdot (1 + 0.833^2) = 0.4038 \n$$

(c)

States that only one node actually does processing are:

$$
\pi(2.0, 0) + \pi(1.1, 0) + \pi(0.1, 1) = 0.1060 + 0.2383 + 0.1656 = 0.5099 \text{ from 1* approach. On the other hand, from 2* approach,} \n$$

$$
\pi(2.0, 0) + \pi(0, 2, 0) + \pi(0, 0, 2) = 0.2384(0.667^2 + 1 + 0.833^2) = 0.5097 \n$$

We can see both approaches produce same results.

**Problem B**

$$
Pr(W = t) = \sum_{n=0}^{\infty} Pr(W = t \mid X = n) \cdot Pr(X = n) \n$$

where $X$ is the number of customers in the system when a new customer is arriving.

Since service time is exponential, when a new customer joins the queue, the current service can be renewed like starting over because of memoryless property.

$$
\rightarrow Pr(W = t \mid X = n) = Exp(n + 1, \mu) = \frac{\mu^{n+1} \cdot t^n \cdot e^{-\mu t}}{n!} \n$$

Note that in order to get through the entire system, $n + 1$ processes should be repeated, including the new coming customer itself.

$$
Pr(W = t) = \sum_{n=0}^{\infty} \frac{\mu^{n+1} \cdot t^n \cdot e^{-\mu t}}{n!} \cdot \alpha_n = \sum_{n=0}^{\infty} \frac{\mu^{n+1} \cdot t^n \cdot e^{-\mu t}}{n!} \cdot \beta^n (1 - \beta) \n$$

$$
\beta = \mu (1 - \beta) \cdot e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\mu \cdot t \cdot \beta)^n}{n!} = \mu (1 - \beta) \cdot e^{-\mu t} \cdot e^{\mu \cdot \beta} \n$$
Problem B

From the MTA of the M/G/1 queue performed in class, we know that

\[ L = 2W = 2(W_q + E(S)) = 2W_q + 2E(S) = \]

\[ = \frac{1 + \lambda \sigma^2}{2} \frac{1}{1 - \rho} E[S] + \rho = \frac{1 + \lambda \sigma^2}{2} \frac{\rho^2}{1 - \rho} + \rho \]

Also, as discussed in class,

\[ L = T(2) |_{z = 1} \]

Hence, our task is to show that

\[ T(2) |_{z = 1} = \frac{1 + \lambda \sigma^2}{2} \frac{\rho^2}{1 - \rho} + \rho \]  

(1)

We also know that

\[ T(2) = (1 - \rho) \frac{(1 - z) K(2)}{K(2) - z} \frac{K(z) - z K(2)}{K(z) - z^2} (1 - \rho) \]

where \( K(z) \) is the probability generating function for the distribution characterizing the number of arrivals within a service time period.

From (2),

\[ T(2) = (1 - \rho) \frac{(K(z) - z^2) [K'(z) - K(z) - z K'(z)] - (K(z) - z K(2)) K''(z)}{(K(z) - z)^2} \]

\[ = (1 - \rho) \frac{K(z) - K^2(z) - (z - z^2) K'(z)}{(K(z) - z)^2} \]  

(3)
From (3), taking into consideration that $K(2) = 1$, we see that $\Pi(2)|_{z=1}$ takes the undefined form $\infty$. So, we proceed by applying L'Hopital's rule (twice) on the quantity:

$$A = \frac{K'(z) - K^2(z) - (2 - z^2)K''(z)}{(K(z) - 2)^2}$$

From the first application of the rule we get:

$$\lim_{z \to 1} A = \lim_{z \to 1} \frac{K'(z) - K(z)K'(z) - (2 - z^2)K''(z)}{2(K(z) - 2)(K'(z) - 1)}$$

$$= \lim_{z \to 1} \frac{[2z - 2K(z)]K'(z) - (2 - z^2)K''(z)}{2(K(z) - 2)(K'(z) - 1)}$$

From the second application of the rule we have:

$$\lim_{z \to 1} A = \lim_{z \to 1} \frac{2[1 - K'(z)]K''(z) + 2[2 - K(z)]K'''(z) - (1 - 2z)K'''(z)}{2(K(z) - 1)(K'(z) - 1) + (K(z) - 2)(2K''(z))}$$

$$= \frac{2[1 - \rho] \rho + K'''(1)}{2(\rho - 1)^2} = \frac{K'''(1)}{2(1 - \rho)} + \frac{\rho}{1 - \rho}$$

So, from (3) and (4) we get:

$$\Pi(2)|_{z=1} = \frac{K'''(1)}{2(1 - \rho)} + \rho$$

In the above, we have used the fact that $K(z)|_{z=1} = \rho$.

That was established in class.
To proceed from (5), we need to express $W'(1)$ in terms of the problem parameters. For this, we proceed as follows:

We know that

$$K(t) = \sum_{i=0}^{\infty} K_i \cdot e^{-\gamma t} i \cdot \gamma$$

so

$$\frac{d^2 K(t)}{dt^2} = \sum_{i=2}^{\infty} i(i-1) \cdot K_i \cdot e^{-\gamma t} \cdot \gamma^2$$

where $A$ denotes the random arrival rate over a service time period.

In class, we argued that $E(C_A) = 2E(S) = \rho$.

A formal proof for this result is by taking conditional expectations:

$$E(C_A) = E[E(C_A | S)] = E[S] = 2E(S) = \rho$$

To compute the quantity $Var(A)$ that appears in (6), we use the result of Eq. (5.6) in the provided note, the M/M/1 queue (also, see next page):

$$Var(A) = \rho + 2\sigma^2$$

Then, we have

$$\frac{d^2}{dt^2} K(t) = \rho + \sigma^2 + \rho - \rho = \rho + \sigma^2$$

and (7) and (15) imply

$$\Pi'(2) |_{\gamma=1} = \frac{1}{\gamma^2} \cdot \frac{\rho}{1 - \rho}$$

which proves (1).
Proving Eq 5.6

\[ \text{Var}(A) = E(A^2) - E^2(A) = \]

\[ = E\left[ E(A^2|S) \right] - E^2\left[ E(A|S) \right] = \]

\[ = E\left[ \text{Var}(A|S) + E^2(A|S) \right] - E^2\left[ E(A|S) \right] = \]

\[ = E\left[ \text{Var}(A|S) \right] + E\left[ E^2(A|S) \right] - E^2\left[ E(A|S) \right] = \]

\[ = E\left[ \text{Var}(A|S) \right] + \text{Var}\left[ E(A|S) \right] = \]

\[ = E(C(S) + \text{Var}(C(S) = \]

\[ = E(C(S) + 2^q \text{Var}(C(S) = \]

\[ = \mu + 2^q \sigma^2. \]
The situation described in the problem can be depicted as follows:

\[ T_1 \]  
\[ T_2 \]

\[ \bar{v}_a = 0.5 \text{ min}^{-1} \]

Where:

- \( T_1 \) = the random processing time at station 1, distributed according to a distribution with mean equal to 1 min and st. dev equal to 0.5 min.

- \( T_2^n \) = the random processing time of a non-defective part at station 2, distributed according to a distribution with mean equal to 1 min and st. dev equal to 0.5 min.

- \( T_2^d \) = the processing time of a defective part at station 2.

This time is further decomposed to

\[ T_2^d = T_{21}^d + T_{22}^d \]

Where \( T_{21}^d \) represents the random time spent in the preparing stage, coming from a distribution with a mean equal to 0.5 min and st. dev equal to 1.5 min, and \( T_{22}^d \) represents the random time necessary for the main processing of the part at station 2, coming from a distribution with mean equal to 1 min and st. dev. equal to 0.5 min (in fact, the same distribution that applies for \( T_2^n \)).

- \( P_d = 0.15 \) = the probability that a part is defective, after being processed at station 1.
Then,

(i) We need $u_1 < 1$ and $u_2 < 1$ for stability.

\[
\begin{align*}
    u_1 &= R_a \mathbb{E}[T_1] = 0.5 \text{ min}^{-1}, \quad 1 \text{ min} = 0.5 < 1 \\
    u_2 &= R_a \mathbb{E}[T_2] = R_a \mathbb{E}[T_2^x] (1 - \beta_d) + R_a \mathbb{E}[T_2^d] \beta_d = \\
    &= R_a \left\{ \mathbb{E}[T_2^x] (1 - \beta_d) + \left( \mathbb{E}[T_2^d] + \mathbb{E}[T_2^d] \right) \beta_d \right\} = \\
    &= 0.5 \text{ min}^{-1} \left\{ \beta_d \left[ 1 - \beta_d (1 - 0.85) \right] \right\} \\
    &= 0.6875 < 1
\end{align*}
\]

So, the line is stable.

(ii) From the previous calculation, we see that:

\[
\begin{align*}
    u_2 &= R_a \left\{ \mathbb{E}[T_2^x] (1 - \beta_d) + \left( \mathbb{E}[T_2^d] + \mathbb{E}[T_2^d] \right) \beta_d \right\} = \\
    &= 0.5 \text{ min}^{-1} \left\{ \beta_d \left[ 1 - \beta_d (1 - 0.85) \right] \right\}
\end{align*}
\]

For stability, we need: $u_2 < 1 \Rightarrow$

\[
\begin{align*}
    0.5 \left[ 1 + 2.5 \beta_d \right] < 1 \Rightarrow \beta_d < 0.4
\end{align*}
\]

(iii) $C_T = C_T^1 + \mathbb{E}[T_1] + C_T^2 + \mathbb{E}[T_2]$

\[
\begin{align*}
    C_T^1 &= \frac{C_0^1 + C_0^1}{2} \frac{u_1}{1 - u_1} \mathbb{E}[T_1] = \frac{0 + 0.5}{2} \frac{0.5}{1 - 0.5} \beta_d = 0.125 \text{ min} \\
    C_T^2 &= \frac{C_0^2 + C_0^2}{2} \frac{u_2}{1 - u_2} \mathbb{E}[T_2] \\
    C_0^2 &= C_d \approx u_1^2 C_0^2 + (1 - u_1^2) C_0^2 = 0.5 \cdot 0.5 = 0.25 \\
    \mathbb{E}[T_2] &= (1 - \beta_d) \mathbb{E}[T_2^x] + \beta_d \mathbb{E}[T_2^d] = \\
    &= 0.85 \text{ min} + 0.15 (2.5 \text{ min} + 1 \text{ min}) = 1.375 \text{ min}.
\end{align*}
\]
To get $C^2_{b_0}$, we work as follows:

$$C^2_{b_0} = \frac{\text{Var}[T_2]}{E[T_2]^2}$$

$$\text{Var}[T_2] = E[T_2^2] - (E[T_2])^2$$

$$E[T_2^2] = (1-p_d) \cdot E[(T_2^n)^2] + p_d \cdot E[(T_2^d)^2]$$

$$E[(T_2^n)^2] = \text{Var}[T_2^n] + (E[T_2^n])^2 = 0.5^2 + 1 = 1.25 \text{ min}^2$$

$$E[(T_2^d)^2] = \text{Var}[T_{21} + T_{22}] + (E[T_{21} + T_{22}])^2 = \text{Var}[T_{21}] + \text{Var}[T_{22}] + (E[T_{21}] + E[T_{22}])^2 = 1.5^2 + 0.5^2 + (2.5 + 1)^2 = 14.75 \text{ min}^2$$

Hence,

$$E[T_2^2] = 0.85 \cdot 1.25 + 0.15 \cdot 14.75 = 3.275 \text{ min}^2$$

$$\text{Var}[T_2] = 3.275 - 1.375^2 = 1.384 \text{ min}^2$$

$$C^2_{b_0} = \frac{1.384 \text{ min}^2}{1.375^2 \text{ min}^2} = 0.739$$

and

$$C_{-1}^2 = \frac{0.025 + 0.722}{2} = \frac{0.6875}{1 - 0.6875} = 1.375 = 1.2 \text{ min}$$

Finally,

$$CT = 0.125 + 1 + 1.2 + 1.375 = 3.7 \text{ min}$$

(iv) For this class of jobs:

$$CT'' = (C_{-1} + E[T_1]) + C_{-1}^2 + E[T_2^n] = 0.125 + 1 + 1.2 + 1 = 3.375 \text{ min}$$
To answer the posed question, we need first to compute the mean effective processing time. For this, we have

\[ E[\text{Teff}] = E[\text{Tp} + \text{Te}] = E[\text{Tp}] + E[\text{Te}] \]

\[ E[\text{Tp}] = 2 \text{ min} \]

\[ E[\text{Te}] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \]

\[ = \frac{1}{0.2 \text{ min}^{-1}} + \frac{1}{0.1 \text{ min}^{-1}} = 5 \text{ min} \]

\[ E[\text{Te} | \text{only part 1 defective}] = \frac{1}{\lambda_1} = \frac{1}{0.2 \text{ min}^{-1}} = 5 \text{ min} \]

\[ E[\text{Te} | \text{only part 2 defective}] = \frac{1}{\lambda_2} = \frac{1}{0.1 \text{ min}^{-1}} = 10 \text{ min} \]

\[ E[\text{Te} | \text{both parts defective}] = \]

\[ = E[\max(\text{Te}, \text{Te}^2) | \text{both parts defective}] \]

\[ = E[\max(\text{Te}, \text{Te}^2) | \text{both parts defective} \wedge \text{Te} < \text{Te}^2] \]

\[ = \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \]

\[ = \frac{1}{\lambda_1 + \lambda_2} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \]

\[ = \frac{1}{0.2 \text{ min}^{-1} + 0.1 \text{ min}^{-1}} \left( 5 + 10 \right) = 11.67 \text{ min} \]
Hence, \( E[\text{Trend}] = 5 \cdot 0.8 - 0.8 + 10 \cdot 0.7 \cdot 0.2 + 11 \cdot 0.3 \cdot 0.2 \approx 2.3 \text{ min} \)

and

\( E[\text{Tsys}] = 2 + 2.3 = 5.3 \text{ min} \)

Therefore, the effective prooing capacity is \( 60 / 5.3 \approx 11.32 \text{ units/hr} \).
Problem 4 (20 points): A service station is processing the commingled stream of two part types, each of which arrives according to a Poisson process with rate $\lambda_i$. Parts are processed on a FCFS basis, and the expected service time for either part is equal to $t_p$ time units, but when switching from one part type to the other, there is an additional deterministic set-up time equal to $t_s$ time units. Provide the stability condition for this station; your response must expressed in terms of the data set provided above.

The probability in a setup between two consecutive jobs can be written as:

$$P(\text{setup}) = P(\text{previous job is type 1}) \cdot P(\text{setup | prev. job type 1}) + P(\text{previous job is type 2}) \cdot P(\text{setup | prev. job type 2})$$

Taking into consideration the Poisson and independent nature of the two arrival processes, we get:

$$P(\text{setup}) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{2}{\lambda_1 + 2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{2}{\lambda_1 + 2} \frac{2}{\lambda_1 + 2} \frac{2}{\lambda_1 + 2} = \frac{22}{(\lambda_1 + \lambda_2)^2}$$

Hence, the expected proc time for any part, when accounting for the potential setups, is:

$$t_e = t_p + P(\text{setup}) t_s = t_p + \frac{22}{(\lambda_1 + \lambda_2)^2} t_s$$

For stability, we need:

$$(\lambda_1 + \lambda_2) t_e < 1 \implies (\lambda_1 + \lambda_2) t_p + \frac{22}{\lambda_1 + \lambda_2} t_s < 1$$
Problem 2 (20 points): A machine can experience two types of failure. Both types of failure can occur only when the machine is operational (i.e., failures are “operation-driven” and not “time-driven”), they occur independently from each other, and their occurrences follow Poisson processes with corresponding rates $\lambda_i$, $i = 1, 2$. Also, the corresponding MTTRs (mean time to repair) are equal to $t_i$, $i = 1, 2$. Answer the following questions:

i. What is the availability of this machine?

ii. If both types of failure are non-destructive and the “nominal” processing times (i.e., the times that are required for the processing of the parts without accounting for the downtimes due to failures) for this machine are uniformly distributed over the interval $[a, b]$, what is the expected number of failures that take place during the processing of a single part?

In your response, consider that all the referred quantities are given in consistent units.

(i) We know that $A = \frac{MTTF}{MTTF + MTTR}$

where
- $MTTF = \text{mean time to failure}$
- $MTTR = \text{mean time to repair}$

Since each failure type occurs according to a Poisson distribution (while the machine is operational) and these two processes are mutually independent,

$MTTF = \frac{1}{\lambda_1 t_1}$

On the other hand

$MTTR = \frac{\lambda_1}{\lambda_1 t_1 + \lambda_2}{t_1} + \frac{\lambda_2}{\lambda_1 t_1 + \lambda_2}{t_2} = \frac{\lambda_1 t_1 + \lambda_2 t_2}{\lambda_1 t_1 + \lambda_2}$

Finally

$A = \frac{\sqrt{(\lambda_1 + \lambda_2)}}{\sqrt{(\lambda_1 t_1 + \lambda_2 t_2)} + \frac{\lambda_1 t_1 + \lambda_2 t_2}{\lambda_1 t_1 + \lambda_2}} = \frac{1}{1 + \frac{\lambda_1 t_1 + \lambda_2 t_2}{\lambda_1 t_1 + \lambda_2}}$
(ii) Let r.v. $T$ be the past processing time

and r.v. $N$ be the number of failures that take place during the past processing.

Then,

$$E(N) = E\left( E\left[ N | T \right] \right) = E\left( (3 + 2) T \right) = (3 + 2) \frac{ab}{2}$$