Problem 1 (20%) There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

Let us denote the various events of interest as follows:

\[ 2H = \text{the flipped coin was two-headed} \]
\[ F = \text{fair} \]
\[ B = \text{biased to come up heads 75\% of the time} \]
\[ H = \text{the coin comes up heads} \]

Then,

\[
P(2H|H) = \frac{P(2H,H)}{P(H)}
\]

\[
= \frac{P(H|2H) \cdot P(2H)}{P(H|2H) \cdot P(2H) + P(H|F) \cdot P(F) + P(H|B) \cdot P(B)}
\]

\[
= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3}}
\]

\[
= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{6} + \frac{3}{4} \cdot \frac{1}{3}}
\]

\[
= \frac{4}{9}
\]
Problem 2 (20%) The number of customers entering a store on a given day is Poisson distributed with mean $\lambda = 10$. The amount of money spent by a customer is uniformly distributed over (0,100). Find the mean and variance of the amount of money that the store takes in on a given day.

Let

$N =$ the number of customers entering the store during a day

$X_i =$ the money spent by the $i$-th customer

$S =$ the money collected by the store on that day

Then

$S = \sum_{i=1}^{N} X_i$

and it is a compound Poisson r.v. (since $N$ is Poisson-distributed).

Hence, according to the results presented in class,

$E(S) = E(N) \cdot E(X) = \lambda \cdot E(X) = 10 \cdot 50 = 500$

$Var(S) = Var(N) \cdot E(X^2)$

$E(X^2) = \frac{1}{100} \int_{0}^{100} x^2 \, dx = \frac{100^3}{3 \cdot 100} = \frac{10^4}{3}$

and

$Var(S) = \frac{10^5}{3}$
Problem 3 (20%) In a production lot of 100 units there are 8 defective ones. We take a sample of 5 units from this lot.

i. What is the probability that there are exactly 3 defective parts in this sample?

ii. What is the probability that there are exactly 3 non-defective parts in this sample?

iii. What is the probability that there are at least 3 non-defective parts in this sample?

\[
\begin{align*}
\text{(i)} & \quad \frac{\binom{92}{2} \times \binom{8}{3}}{\binom{100}{5}} \\
\text{(ii)} & \quad \frac{\binom{92}{3} \times \binom{8}{2}}{\binom{100}{5}} \\
\text{(iii)} & \quad \frac{\binom{92}{3} \times \binom{8}{2} + \binom{92}{4} \times \binom{8}{1} + \binom{92}{5} \times \binom{8}{0}}{\binom{100}{5}}
\end{align*}
\]

Remark: Notice that in case (i)

\[
\frac{\binom{92}{2} \times \binom{8}{3}}{\binom{100}{5}} = \frac{92! \times 8!}{8! \times 90!} \times \frac{3! \times 5!}{5! \times 95!} = \frac{5!}{2! \times 3!} \times \frac{91 \times 92 \times 6 \times 7 \times 8}{90 \times 96 \times 97 \times 98 \times 96}
\]

This last expression can be interpreted as:

\[
\frac{8}{100} \cdot \frac{7}{99} \cdot \frac{6}{98} \cdot \frac{2}{97} \cdot \frac{1}{96} + \cdots
\]

\[\left(\frac{5}{3}\right)\] = 10 terms in total, each equal to \[\frac{91 \times 92 \times 6 \times 7 \times 8}{90 \times 96 \times 97 \times 98 \times 96}\] Each of these terms expresses the probability of getting the defect in certain positions of the sampling sequence.
Problem 4 (20%)

i. Show that the moment generating function of a random variable \( X \) modeling a Bernoulli trial with success probability \( p \) is equal to \( \phi_1(t) = pe^t + 1 - p \).

ii. Use the result in part (i) above, in order to show that the moment generating function of a Binomial distribution with parameters \( n \) and \( p \) is equal to \( \phi_2(t) = (pe^t + 1 - p)^n \).

iii. Use the result of part (ii) above, in order to show that if \( X \) and \( Y \) are independent r.v.'s following Binomial distributions with respective parameters \( (n_X, p) \) and \( (n_Y, p) \), then the r.v. \( X + Y \) follows a Binomial distribution with parameters \( (n_X + n_Y, p) \).

iv. What is the intuitive interpretation of the result established in part (iii) above?

\[
(i) \quad \phi_1(t) = E[e^{tX}] = pe^t + (1-p)e^0 = pe^t + 1 - p \\
(ii) \quad \text{Let } Y \sim \text{Binomial}(n, p) \text{ then } \\
Y = \sum_{i=1}^{n} X_i \text{ where } X_i, i = 1, \ldots, n, \text{ are iid r.v.'s distributed according to a Bernoulli distribution with success probability equal to } p. \\
\text{Hence } \phi_Y(t) = (pe^t + 1 - p)^n \\
(iii) \quad \phi_{X+Y}(t) = E[e^{t(X+Y)}] = E[E[e^{tX}] E[e^{tY}]] = (pe^t + 1 - p)^{n_X} (pe^t + 1 - p)^{n_Y} = (pe^t + 1 - p)^{n_X + n_Y} \\
\Rightarrow X + Y \sim \text{Binomial}(n_X + n_Y, p) \\
(iv) \quad X + Y \text{ essentially models the number of successes in } (n_X + n_Y) \text{ Bernoulli trials with success prob. equal to } p.
Problem 5 (20%) Let $X$ be exponential with mean $1/\lambda$; that is

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty$$

i. Argue that

$$f_{X|X>1}(x) = \frac{\lambda e^{-\lambda x}}{P(X > 1)}, \quad 1 < x < \infty$$

ii. Use the result of part (i) to show that $E[X|X > 1] = 1 + \frac{1}{\lambda}$

iii. Can you provide an intuitive interpretation for the result of part (ii)?

\[
\begin{align*}
\{x \in \mathbb{R}, X \leq x + dx, X > 1 \} &= \int_{1}^{\infty} f_{X|X>1}(x) dx \\
&= \int_{1}^{\infty} \frac{\lambda e^{-\lambda x}}{P(X > 1)} dx \\
&= \frac{\lambda e^{-\lambda x}}{P(X > 1)} dx \int_{1}^{\infty} \frac{dx}{P(X > 1)} \\
&= \left\{ \begin{array}{ll}
0 & , \quad x < 1 \\
\frac{1}{\lambda} & , \quad x > 1
\end{array} \right.
\end{align*}
\]
A more formal argument for part (i) is as follows:

\[ F_{|X > 1}(x) = P \{ X < x \mid X > 1 \} = \]
\[ = \begin{cases} 
\frac{P \{ 1 < X < x \}}{P \{ X > 1 \}}, & x > 1 \\
0, & \text{otherwise}
\end{cases} \]

But

\[ \frac{P \{ 1 < X < x \}}{P \{ X > 1 \}} = \frac{P \{ X > 1 \} - P \{ X > x \}}{P \{ X > 1 \}} = \]
\[ = 1 - \frac{e^{-2x}}{P \{ X > 1 \}} \]

Finally

\[ f_{|X > 1}(x) = \frac{dF_{|X > 1}(x)}{dx} = \begin{cases} 
\frac{2e^{-2x}}{P \{ X > 1 \}}, & x > 1 \\
0, & \text{otherwise}
\end{cases} \]
(ii) $P(x>1) = \int_1^{\infty} x e^{-2x} \, dx = 2 \int_1^{\infty} e^{-2x} \, dx = 2 \left[ -\frac{1}{2} e^{-2x} \right]_1^{\infty} = 2 \left( 0 - \frac{1}{2} e^{-2} \right) = e^{-2}$

Then,

$E[x \mid x>1] = \int_1^{\infty} x P(x \mid x>1) \, dx = 2 e^{-2} \int_1^{\infty} e^{-2x} \, dx = 2 e^{-2} \left[ -\frac{1}{2} e^{-2x} \right]_1^{\infty} = 2 e^{-2} \left( \frac{1}{2} - \frac{1}{2} e^{-2} \right) = 1 + \frac{1}{2}$

(iii) The result of part (iii) essentially implies that the expected value of the conditional distribution is equal to the expected value of the (unconditional) distribution plus the lower bound for the value of $x$ established by the conditioning. In other words,

$E[x \mid x>1] = \frac{1}{2} = E[x]$

The above equation is a manifestation of the "memoryless" property of the exponential distribution.