HW 2 Solution

1. (16 pts)

D = 5000/yr, C = 600/unit, 1 year = 300 days, i = 0.06, A = 300

Current ordering amount Q = 200

(a) \[ T^* = \frac{Q}{D} \times 300 \text{days} = \frac{200}{5000} \times 300 = 12 \text{ days} \]

(b) Total (Holding + Setup) cost would be

\[ TC = \frac{iC}{2} Q + \frac{D}{Q} A = \frac{0.06 \times 600}{2} \times 200 + \frac{5000}{200} \times 300 = 11,100/\text{yr} \]

(c) The optimum cost would be

\[ \sqrt{2ADh} = \sqrt{2 \times 300 \times 5000 \times 0.06 \times 600} = 10392.30/\text{yr} \]

(d) \( T^* \) is 12 days. The closest power of two is 16 days (16/300 yr).

\[ TC(16 \text{ days}) = \frac{TDh}{2} + \frac{A}{T} = \frac{16}{2} \times 5000 \times 0.06 \times 600 \times 2 + \frac{300}{16} \times 300 = 10425/\text{yr} \]

The power of two on the other side of 12 days is 8 days (8/300 yr).

\[ TC(8 \text{ days}) = \frac{TDh}{2} + \frac{A}{T} = \frac{8}{2} \times 5000 \times 0.06 \times 600 \times 2 + \frac{300}{8} \times 300 = 13650/\text{yr} \]

2. (17 pts)

D = 200/month = 2400/yr, A = (100+55)*1.5 = 232.5

P = 50/hr = 50*6*20*12/yr = 72000/yr, i = 0.22, C = 2.50

(a) \[ Q^* = \sqrt{\frac{2AD}{h(1-\frac{P}{D})}} = \sqrt{\frac{2 \times 232.5 \times 2400}{0.22 \times 2.50 \times (1-\frac{2400}{72000})}} = 1448.8 \approx 1449 \]

(b) \( H = Q^* \left(1 - \frac{D}{P}\right) = 1449 \times \left(1 - \frac{2400}{72000}\right) = 1400.7 \approx 1401 \)

(c) \[ \frac{D}{P} = \frac{2400}{72000} = 0.0333 = 3.33\% \]
The answer for part (c) above does not account for the set up time as part of the “working time” for the production of the considered filters. If we want to include this additional time in our answer, we can work as follows:

The time for producing a single lot, including the relevant set time is:
\[ \frac{1449}{50} + 1.5 = 30.48 \text{ hrs.} \]

The number of lots produced in a year is: \[ \frac{2400}{1449} = 1.656. \]

Hence, the total production time spend on this item in a year is:
\[ 50.48 \times 40.48 = 50.48 \text{ hrs} \]

According to the problem, a year has \( 6 \times 20 \times 12 = 1440 \) working hours.

Hence, the percentage of working time taken by the production of this item is;
\[ \frac{50.48}{1440} = 0.0350, \text{ i.e.,} \, 3.50\%, \]
which is greater than the original response, since it accounts also for the setup time. The increase is not significant, but this is due to the selection of \( Q^* \) as discussed above.

3. (17 pts)

(a)

EOQ of A:
\[ \sqrt{\frac{2 \times 100 \times 20000}{0.2 \times 2.5}} = 2828.42 \approx 2828 \Rightarrow Q_A = 2828 \]

EOQ of B:
\[ \sqrt{\frac{2 \times 100 \times 20000}{0.2 \times 2.4}} = 2886.75 \approx 2887 \Rightarrow Q_B = 3000 \]

EOQ of C:
\[ \sqrt{\frac{2 \times 100 \times 20000}{0.2 \times 2.3}} = 2948.84 \approx 2949 \Rightarrow Q_C = 4000 \]
Therefore, optimal order quantity is 4000 with source C.

(b)  
Holding + Setup cost = 100 \times \frac{20000}{4000} + 0.2 \times 2.3 \times \frac{4000}{2} = 1420

(c)  
Cycle Time = \frac{4000}{20000} = 0.2 \text{ year} = 2.4 \text{ months.}  
Replenishment lead time = 3 \text{ months.}  
Reorder point = \frac{3}{2.4} \times 4000 = 5000 \rightarrow 1000 \text{ units is reorder point}

It is interesting to interpret the above result for part (c) in terms of the definition of the Inventory Position IP(t) that will be introduced in class in the upcoming lectures. We shall define

\[ IP(t) = OHI(t) + O(t) - BO(t) \]  

where

- \( OHI(t) \) denotes the on-hand-inventory at time \( t \);
- \( O(t) \) denotes the “pipeline” inventory at time \( t \) (i.e., material ordered but not received yet);
- \( BO(t) \) denotes the backorders at time \( t \).
In the setting of this problem, BO(t) = 0 for all t, since we do not allow any backorders. But then, the above picture implies that ROP = ID = (3/12) \times 20000 = 5000 with respect to the inventory position IP(t) and not the on-hand-inventory.

4. (17 pts)
We are given the following information, annual demand = 140 units, ordering cost = 30 per order, holding cost = 18%, and cost function

\[ C(Q) = \begin{cases} 
350Q & : 1 \leq Q \leq 25, \\
8750 + 315(Q - 25) & : 26 \leq Q \leq 50, \\
16625 + 285(Q - 50) & : 51 \leq Q. 
\end{cases} \]

Then we have,

\[ \frac{C(Q)}{Q} = \begin{cases} 
350 & : 1 \leq Q \leq 25, \\
315 + \frac{875}{Q} & : 26 \leq Q \leq 50, \\
285 + \frac{2375}{Q} & : 51 \leq Q. 
\end{cases} \]

- There is a typo in the first line of the right-hand-side in the above equation: the correct expression is 350 and not 350Q.

The total annual cost function, G(Q), that is implied by the above average unit costs, is given by:

\[ G(Q) = \frac{DC(Q)}{Q} + AD + h \left( \frac{C(Q)}{Q} \right) Q. \]

The Q's that minimize this last function for each of the three expressions of C(Q)/Q can be obtained by substituting each of these three expressions in G(Q) and computing the minimum of the resulting function. This procedure gives us:

\[ Q(1) = 12 \]
\[ Q(2) = 67 \]
\[ Q(3) = 115. \]

Observing that Q(2) does not fall into the correct interval, we replace it with Q'(2) =
50, which is the closest point of the corresponding valid interval to 67. Then, we get:

\[
G(Q(1)) = (350)(140) + \frac{(30)(140)}{12} + \frac{(0.18)(350)(12)}{2} = 49728
\]

\[
Q(2)) = (315 + 875/50)(140) + (30)(140)/50 + [(0.18)(315+875/50)(50)]/2 = 48130.25
\]

\[
G(Q(3)) = \left(285 + \frac{2375}{115}\right)(140) + \frac{(30)(140)}{115} + \frac{(0.18)\left(285 + \frac{2375}{115}\right)(115)}{2} = 45991
\]

Since Q = 115 results in a lower cost, company Y should use an order size of 115 units.

**Remark:** In fact, in the case of incremental quantity discounts, it can be shown that the cost curves \(G(Q)\) with a minimizing value \(Q(i)\) that does not belong in the corresponding price interval, need not be considered any further when searching for the best Q value. This can be seen when considering the way that the various cost functions \(G(Q)\) intersect in the case of incremental discounts (c.f. the figure in the material that was posted in the electronic reserves). Hence, in the above problem, we could have skipped the evaluation of \(G(Q(2))\).

**5. (17 pts)**

Order quantity given data

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>12,500</td>
<td>15,000</td>
<td>15,000</td>
</tr>
<tr>
<td>A</td>
<td>150</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>h</td>
<td>2.4</td>
<td>3.5</td>
<td>3</td>
</tr>
<tr>
<td>Unit. Stor(f)</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>EOQ</td>
<td>1250</td>
<td>828.0786712</td>
<td>894.427191</td>
</tr>
<tr>
<td>Stor. Need</td>
<td>6250</td>
<td>3312.314685</td>
<td>3577.708764</td>
</tr>
<tr>
<td>Total Storage</td>
<td>13140.02345</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Since total required storage area is over 6000 sq. ft., we need to adjust order quantities. We can find the optimal order quantities through the search process over the Lagrange multiplier $\lambda$, discussed in class, that computes the values $Q_i = \sqrt{\frac{2A_iD_i}{h_i+2\lambda f_i}}$ and checks whether they satisfy the resource constraint as equality.

After some search on the values of $\lambda$, we get: $\lambda^* = 1.204799$

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
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</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>12,500</td>
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<td>80</td>
</tr>
<tr>
<td>h</td>
<td>2.4</td>
<td>3.5</td>
<td>3</td>
</tr>
<tr>
<td>Unit. Stor($)</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>NEW Q</td>
<td>509.4621425</td>
<td>427.3999954</td>
<td>435.7723896</td>
</tr>
<tr>
<td>Stor. Need 2</td>
<td>2547.310713</td>
<td>1709.599982</td>
<td>1743.089559</td>
</tr>
<tr>
<td>Total Storage</td>
<td></td>
<td>6000.000253</td>
<td></td>
</tr>
</tbody>
</table>

Thus, the optimized order quantities for item 1, 2, and 3 should be 509, 427, and 436, respectively. As discussed in class, $-\lambda^*$ denotes the derivative of the optimal cost with respect to the size of the storage area $F$, and therefore, we should not be willing to pay more than 1.2 dollars per extra sq. ft.

6. (16 pts)

<table>
<thead>
<tr>
<th>Time</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>100</td>
<td>150</td>
<td>75</td>
<td>75</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>Inventory on hand</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Actual Demand</td>
<td>40</td>
<td>150</td>
<td>75</td>
<td>75</td>
<td>50</td>
<td>60</td>
</tr>
</tbody>
</table>
In the cost calculations provided in the following table, each cell \((i,j), j \in \{1,\ldots,6\}, i \in \{1,\ldots,j\}\), denotes the cost of the plan that orders for the demand of period \(j\) at period \(i\), while following an optimal order plan over the periods \(1,\ldots, i-1\).

<table>
<thead>
<tr>
<th>Time</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost Calculation</td>
<td>80</td>
<td>192.5</td>
<td>305</td>
<td>473.75</td>
<td>623.75</td>
<td>848.75</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>216.25</td>
<td>328.75</td>
<td>441.25</td>
<td>621.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>240</td>
<td>296.25</td>
<td>371.25</td>
<td>506.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>296.25</td>
<td>333.75</td>
<td>423.75</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>376.25</td>
<td>421.25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>413.75</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Order Quantity</td>
<td>40</td>
<td>225</td>
<td>125</td>
<td></td>
<td>60</td>
<td></td>
</tr>
</tbody>
</table>

As some concrete examples, first consider the cost of 192.5 in cell \((1,2)\). This is the cost of ordering in period 1 the total demand for the first two periods, \(D1+D2\). Hence the total cost is the ordering cost plus the inventory holding cost of carrying \(D2\) forward by one period, i.e., \(80+0.75\times150 = 192.5\). On the other hand, cell \((3,5)\) corresponds to the decision of ordering the demand of the last period in that sub-problem (i.e., \(D5\)) in period 3. But then, the Wagner-Whitin property implies that in period 3 I have to order enough units to meet also the demand for periods 3 and 4. Hence, this order will cost me the ordering cost of 80 plus the holding cost of \(0.75\times D4+2\times0.75\times D5\), for a total of 211.25. To this value, I need to add the cost of meeting the demand for the first two periods. But from the above table, I can see that the best way to meet the demand for these two periods will result in a cost of 160 (the minimum value in column 2). Hence, the best possible cost corresponding to the decision \(D5\) in sub-problem 5 of ordering for demand \(D5\) in period 3 is
211.25 + 160 = 371.25 (the value indicated in the corresponding cell (3,5)).

The optimal order plan is obtained by recognizing first that, according to the above table, it is best to order for demand D6 separately in period 6. Hence, in period 6, we need to place an order for 60 units. To deal with the remaining five periods, we need to consult column 5 in the above table, which indicates that the order for demand D5 should be placed in period 4. Hence, we also add an order to our plan for D4 + D5 = 125 units in period 4. To identify the best way to cover the demand for the first three periods, we consult column 3 of the above table. This column suggests that the demand D3 must be ordered in period 2. Hence, we place an order for D2 + D3 = 225 units in our plan. And at this point we are also left with the demand of period 1, which of course must be covered by a separate order in that period.

Finally, we also notice that according to the “Planning Horizon” theorem of the Wagner-Whitin algorithm, we could have skipped the calculation of all the cells in each column $j \in \{1,\ldots,6\}$, that lies above the highlighted cell in column $j-1$, without compromising the identification of the optimal plan (i.e., in columns 3 and 4, we could have skipped the evaluation of their first cells, and in columns 5 and 6 we could have skipped the evaluation of the first three cells).