This test is take home. Please work by yourself. If you have questions, ask me. Turn the test in by 5PM on Thursday, December 9. Good luck!

1. A fax machine receives calls according to a nonhomogeneous Poisson process at the rate of 3/hour between 8:00AM and 10:00AM, and at the rate of 2/hour between 10:00AM and 12:00PM.

(a) What’s the probability that no fax messages are received during the period 9:00AM–11:00AM?

Solution: 0.0067.

(b) Suppose that there were 5 incoming calls between 8:00AM and 9:00AM. If the fax machine was broken between 8:30AM and 9:00AM, what’s the probability that it missed at least one call?

Solution: 0.9688.

2. People arrive at an automatic teller machine according to a Poisson process at the rate of 5/hour. Each arrival is a male with probability 0.4. If the amount of money withdrawn by a customer is exponential with mean $60, find the expected value and variance of the amount of money withdrawn by males in the next 2 hours. (Assume that all random variables are independent.)

Solution: 240, 28800.

3. Suppose $X_1, X_2, \ldots$ are the interarrival times of a nonhomogeneous Poisson process with intensity function $\lambda(t)$.

(a) Are the $X_i$’s independent?

Solution: No.
(b) Are the $X_i$’s identically distributed?

**Solution:** No. □

(c) What’s the distribution of the number of arrivals up to time $t$?

**Solution:** $\text{Pois}(\int_0^t \lambda(s) \, ds)$. □

(d) What’s the distribution of $X_1$?

**Solution:** $\mathbb{P}(X_1 > t) = \mathbb{P}(N(t) = 0) = \exp(- \int_0^t \lambda(s) \, ds)$. □

4. Suppose a Markov chain starts out in its limiting distribution, i.e., $\mathbb{P}(X_m = j) = \pi_j$ for all $j$ and $m$. Prove that $\mathbb{P}(X_m = j|X_{m+1} = i) = \pi_j P_{ji}/\pi_i$.

**Solution:**

$$\mathbb{P}(X_m = j|X_{m+1} = i) = \frac{\mathbb{P}(X_m = j, X_{m+1} = i)}{\mathbb{P}(X_{m+1} = i)} = \frac{\mathbb{P}(X_{m+1} = i|X_m = j)\mathbb{P}(X_m = j)}{\mathbb{P}(X_{m+1} = i)}. □$$

5. Here are some quick random variate questions. Consider the following 12 i.i.d. $\text{Unif}(0,1)$’s.

0.91 0.38 0.46 0.09 0.82 0.93 0.17 0.31 0.47 0.52 0.53

**Note:** Some of these questions have multiple answers!

(a) Use these 12 $\text{Unif}(0,1)$’s and our central limit theorem method to generate a $\text{Nor}(0,1)$ random variate.

**Solution:** 0.21. □

(b) Use your answer in Question 5a to generate a $\text{Nor}(-3, 4)$ random variate.

**Solution:** −2.58. □
(c) Use the first 2 Unif(0,1)’s and the Box-Müller method to produce two Nor(0,1) random variates, $Z_1$ and $Z_2$.

**Solution:** $-0.316, 0.297$. □

(d) Use your answer in Question 5c to generate a $\chi^2(2)$ random variate.

**Solution:** $0.188$. □

(e) Use the first U(0,1) to generate an Exp(1/6) random variate.

**Solution:** $0.566$ or $14.45$. □

(f) Use the first few U(0,1)’s to generate a Pois(1/2) random variate.

**Solution:** $1$. □

(g) Consider the first-order autoregressive process [AR(1)],

$$X_{i+1} = \phi X_i + Z_{i+1},$$

where the $Z_i$’s are i.i.d. Nor($0,1 - \phi^2$) random variables, and $-1 < \phi < 1$. What is $\text{Cov}(X_2, X_6)$?

**Solution:** $\phi^4$. □

(h) Consider the AR(1) process from Question 5g. Further suppose that $\phi = 0.5$ and $X_0 = 3$. Use your answer for $Z_1$ and $Z_2$ from Question 5c to calculate $X_1$ and $X_2$.

**Solution:** $X_1 = 0.5(3) + 0.866(-0.316) = 1.226$. □

6. Here are some questions on simulation output analysis.

(a) Suppose that we calculate $b = 3$ more-or-less i.i.d. normal batch means from a total of $n = 300$ observations:

$$\bar{X}_{1,100} = 3.5 \quad \bar{X}_{2,100} = 5.5 \quad \bar{X}_{3,100} = 1.5$$
Use the method of batch means to obtain an approximate two-sided 95% confidence interval for the mean of the above data. (For your information, $t_{0.025,1} = 12.706$, $t_{0.025,2} = 4.303$, and $t_{0.025,3} = 3.182$.)

Solution: $[-1.5, 8.5]$. □

(b) Suppose I’ve broken my data into 5 batches of 60 observations. Would you expect a shorter or longer confidence interval? (Don’t perform any calculations.)

Solution: Shorter. □

(c) Suppose the batch means variance parameter estimator is exactly chi-squared, i.e., suppose that

$$\hat{V}_B \sim \frac{\sigma^2 \chi^2(b-1)}{b-1}. $$

Find $E[\hat{V}_B^{1/2}]$ (this quantity turns out to be proportional to the expected half-length of the batch means confidence interval).

Solution:

$$\sigma \sqrt{\frac{2}{b-1}} \frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{b-1}{2}\right)}. \quad \square$$

(d) Graph your expression in Question 6c as a function of $b$.

7. Consider the following 12 random variates.

| 9.9 | 3.8 | 4.6 | 0.9 | 8.2 | 9.3 | 6.2 | 1.7 | 3.1 | 4.7 | 5.2 | 5.3 |

Calculate the overlapping batch means estimator for $\sigma^2$ based on a batch size of 6.

Solution: 3.174. □

8. Let’s do a little exercise with antithetic variates.

(a) First of all, suppose that $U_1$ and $U_2$ are i.i.d. Unif(0,1), and define $X_i = \ln(U_i)$ for $i = 1, 2$. Find the expected value and variance of $\bar{X} = (X_1 + X_2)/2$.

Solution: $-1, 1/2$. □
(b) Now suppose \( U \sim \text{Unif}(0, 1) \), and let \( Y_1 = \ell \ln(U) \) and \( Y_2 = \ell \ln(1 - U) \). Use your favorite computer language to generate a scatter plot of 1000 i.i.d. pairs of \((Y_1, Y_2)\).

(c) Continuing with (b), find the expected value and variance of \( \bar{Y} = (Y_1 + Y_2)/2 \).

**Solution:** \(-1, 1 - \frac{\pi^2}{12}\). □

(d) Compare your answers in parts (a) and (c).

**Solution:** \( \mathbb{E}[\bar{X}] = \mathbb{E}[\bar{Y}] \), but \( \text{Var}(\bar{X}) = \text{Var}(\bar{Y}) \). □

9. An old favorite (with a typo fixed). The current price of a stock is 100. The price at time 1 will be either 50, 100, or 200. An option to purchase \( y \) shares of the stock at time 1 for the (present value) price \( ky \) costs \( cy \).

(a) If \( k = 120 \), show that an arbitrage opportunity occurs if and only if \( c > 80/3 \).

(b) If \( k = 80 \), show that there is no arbitrage opportunity if and only if \( 20 \leq c \leq 40 \).

10. Suppose that \( \mathcal{W}(t) \) is standard Brownian motion. Find the mean and variance of

(a) \( \int_0^t s \, d\mathcal{W}(s) \).

**Solution:** \( 0, t^3/3 \). □

(b) \( \int_0^t s^2 \, d\mathcal{W}(s) \).

**Solution:** \( 0, t^5/5 \). □

11. A stock is currently priced at $15 and moves as an exponential Brownian motion with (upward) drift of 10% a year (compounded continuously) and volatility of 25% a year. The current interest rate is 6%. What’s the value of an option on the stock for $18 one year from now?

**Solution:** 0.7886. □