ISyE 6759 — Test 2
Fall 2004

This test is take home. Please work by yourself. If you have questions, ask me. Turn the test in by the beginning of our Thursday, November 11 class. Good luck!

1. Suppose that mistakes in biological cell division occur according to a Poisson process at the rate of 3 per year, and that an individual dies when 200 such mistakes occur.
   
   (a) Find the mean lifetime of an individual.
   (b) Find the variance of the lifetime of an individual.
   (c) Find the approximate probability that an individual dies before age 63.

2. Cars pass a certain location according to a PP(\(\lambda\)). Debby wants to cross the street at that location and waits until she can see that no cars will come by in the next \(y\) time units.
   
   (a) Find the probability that her waiting time is 0.
   (b) Find her expected waiting time. Hint: Condition on the first car.

3. Suppose that \(\{N(t)\}\) is a PP(\(\lambda\)) that’s independent of the i.i.d. sequence \(X_1, X_2, \ldots\), each of which has mean \(\mu\) and variance \(\sigma^2\). Find
   
   \[
   \text{Cov} \left( N(t), \sum_{i=1}^{N(t)} X_i \right).
   \]

4. Using the CLT for suitable Poisson r.v.’s, prove that
   
   \[
   \lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.
   \]

5. Determine the equivalence classes and the periodicity of the various states for Markov chains with the following transition matrices.
   
   (a) \[
   \begin{pmatrix}
   0 & 0 & 1 & 0 \\
   1 & 0 & 0 & 0 \\
   \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
   \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
   \end{pmatrix}.
   \]
6. Let $Y_n$ be the sum of $n$ independent rolls of a fair die. Find
$$\lim_{n \to \infty} P(Y_n \text{ is a multiple of 11}).$$

7. A particle moves among $n+1$ vertices that are situated on a circle in the following manner. At each step, it moves one step either clockwise (with probability $p$) or counterclockwise (probability $q = 1 - p$). Starting at state 0, say, let $T$ be the time until the first return to 0. Find the probability that all states have been visited by time $T$. Hint: Condition on the initial transition and then use results from the gambler’s ruin problem.

8. A total of $m$ red and $m$ green balls are distributed among two urns, with each urn containing $m$ balls. At each stage, we randomly select a ball from each urn and interchange them. Let $X_n$ denote the number of red balls in urn 1 after the $n$th interchange.

(a) Give the transition probabilities of the Markov chain $X_n$, $n \geq 0$.

(b) Without any computations, what do you think are the limiting probabilities of the MC?

(c) A Markov chain is reversible if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j$, where the $\pi_i$’s are the limiting probabilities. Find the limiting probabilities and determine whether or not the stationary MC is reversible.

9. Suppose we can model the difference $Y(t)$ in two stock prices as a BM with variance parameter $\sigma^2 = 4$.

(a) Suppose stock $A$ is ahead of stock $B$ by 3 at the six-month mark. What’s the prob that it’ll also be ahead at $t = 1$ year?

(b) Now suppose that stock $A$ is ahead by 3 at time $t = 1$. What’s the prob that it was ahead at the 6-month mark?

10. Suppose $W(t)$ is standard Brownian motion. Find the expected value and variance of $\int_0^1 W^2(s) \, ds$.

11. Find
$$\mathbb{E} \left[ \int_0^1 W(s) \, ds \right].$$
12. Show that

\[ Y(t) = \exp\left\{ \lambda W(t) - \frac{-\lambda^2 t}{2} \right\} \]

is a martingale.

13. Extra Credit! Simulate Brownian motion from time 0 to time 1, with the intent of estimating \( E[\int_0^1 B(t) \, dt] \), where \( B(t) \) is a standard Brownian bridge process. To do this,

(a) Use Donsker to generate an approximate realization of Brownian motion, \( W(t) \).

(b) Create the approximate bridge, \( B(t) = W(t) - tW(1) \).

(c) Calculate the area under that realization of the bridge.

(d) Repeat many times, and take the average.

If you need a little more help, just ask.