These are Poisson and Markov chain problems from Ross. Do as many as you feel comfortable with.

1. The time $T$ required to repair a machine is exponentially distributed with mean 1/2 (hours).
   (a) What’s the probability that a repair time exceeds 1/2 hour?
   (b) What’s the probability that a repair takes at least 12.5 hours given that its duration exceeds 12?

**Solution:** (a) Since $T \sim \text{Exp}(2)$, we have
   \[ \Pr(T > 1/2) = \int_{1/2}^{\infty} 2e^{-2t} \, dt = e^{-1}. \]
   (b) Using the memoryless property, we have
   \[ \Pr(T > 12.5 | T > 12) = \Pr(T > 1/2) = e^{-1}. \]

2. Suppose you arrive at a single-server bank to find 5 other customers, one being served and the other four waiting in line. You join the end of the line. If the service times are all exponential with rate $\mu$, what’s the expected amount of time you’ll spend in the bank?

**Solution:** Let $T$ denote the time that you’re in the bank; let $S_i$ be the service time of the $i$th person in line ($i = 1, 2, 3, 4$); let $R$ be the remaining service time of the guy in service when you show up; and let $S$ be your service time. Obviously, the $S_i$’s and $S$ are i.i.d. Exp($\mu$). By the memoryless property, so is $R$. Therefore,
   \[ E[T] = E[R + S_1 + S_2 + S_3 + S_4 + S] = 6/\mu. \]

3. Suppose $X$ is exponential. Without any computations, tell which one of the following is correct. Explain.
   (a) $E[X^2 | X > 1] = E[(X + 1)^2]$.
   (b) $E[X^2 | X > 1] = E[X^2] + 1$.
   (c) $E[X^2 | X > 1] = (1 + E[X])^2$. 
Solution: By the memoryless property, the conditional distribution of $X$, given that $X > 1$, is the same as the unconditional distribution of $X + 1$. Therefore, (a) is correct. Since the right-hand sides of (b) and (c) are obviously different than that of (a), we see that (b) and (c) are false. □

4. Consider a post office with two clerks. Three people, A, B, and C, enter at the same time. A and B go directly to the clerks, and C waits until either A or B leaves before he begins service. What’s the probability that A is still in the post office after the other two have left when the service times for each clerk are

(a) exactly (nonrandom) 10 minutes?
(b) i with probability 1/3, $i = 1, 2, 3$?
(c) exponential with rate $\mu$?

Solution: In each case, we want $\Pr(A > B + C)$.

(a) Here, $A = B = C = 10$ implies that $\Pr(A > B + C) = 0$. □

(b) Now, $A, B, C$ are i.i.d. discrete uniform $\{1, 2, 3\}$. This implies that $\Pr(A > B + C) = \Pr(A = 3, B = 1, C = 1) = 1/27$. □

(c) Now, $A, B, C$ are i.i.d. $\text{Exp}(\mu)$. First of all, note that

\[
\Pr(A > B + x) = \int_0^\infty \Pr(A > B + x|B = y) f_B(y) \, dy
= \int_0^\infty \Pr(A > x + y) \mu e^{-\mu y} \, dy
= \int_0^\infty e^{-\mu(x+y)} \mu e^{-\mu y} \, dy
= \frac{1}{2} e^{-\mu x}.
\]

This implies that

\[
\Pr(A > B + C) = \int_0^\infty \Pr(A > B + C|C = x) f_C(x) \, dx
= \int_0^\infty \Pr(A > B + x) \mu e^{-\mu x} \, dx
= \int_0^\infty \frac{1}{2} e^{-\mu x} \mu e^{-\mu x} \, dx
= \frac{1}{4}. \quad \Box
\]
5. The lifetime of a radio is exponential with a mean of two years. If Joey buys a 10-year-old radio, what’s the probability that it will be working an additional 10 years?

**Solution:** By the memoryless property,

\[
\Pr(T > 20 | T > 10) = \Pr(T > 10) = \int_{10}^{\infty} \frac{1}{10} e^{-t/10} \, dt = e^{-1}. \quad \square
\]

6. If \( X \) has failure rate function \( r(t) \), show that \( \mathbb{E}[X] = \mathbb{E}[1/r(X)] \). Hint: Use the Law of the Unconscious Statistician.

**Solution:** Assume that \( X \) is a positive random variable. By the Law of the Unconscious Statistician, we have

\[
\mathbb{E}[1/r(X)] = \int_{\mathbb{R}} \frac{1 - F(x)}{f(x)} f(x) \, dx = \int_{0}^{\infty} \int_{x}^{\infty} f(t) \, dt \, dx = \int_{0}^{\infty} t f(t) \, dt = \mathbb{E}[X]. \quad \square
\]

By the way, the integral interchange a couple of steps up is a nice little trick that allows us to calculate the expected value in a slightly non-standard way.

7. If \( X_i, i = 1, 2, 3 \), are independent exponential RV’s with rates \( \lambda_i, i = 1, 2, 3 \), find

(a) \( \Pr(X_1 < X_2 < X_3) \).

(b) \( \Pr(X_1 < X_2 | \max(X_1, X_2, X_3) = X_3) \).

(c) \( \mathbb{E}[\max X_i | X_1 < X_2 < X_3] \).

(d) \( \mathbb{E}[\max X_i] \).
Solution: (a) We have
\[ \Pr(X_1 < X_2 < X_3) = \Pr(X_1 = \min(X_1, X_2, X_3)) \Pr(X_2 < X_3|X_1 = \min(X_1, X_2, X_3)) \]
\[ = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3}, \]
where the first term follows as usual, and the second follows from the memoryless property. □

(b) Now result (a) gives us
\[ \Pr(X_1 < X_2|X_3 = \max(X_1, X_2, X_3)) \]
\[ = \frac{\Pr(X_1 < X_2 < X_3) + \Pr(X_2 < X_1 < X_3)}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_3} \]
\[ = \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_3}. \]

(c) Let’s do some preliminary stuff first. To begin,
\[ \Pr(X_1 < X_2 < X_3 < w) \]
\[ = \int_0^w \int_0^{x_3} \int_0^{x_2} \lambda_1 \lambda_2 \lambda_3 \exp\{- (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)\} \, dx_1 \, dx_2 \, dx_3 \]
\[ = \frac{\lambda_2 \lambda_3 e^{- (\lambda_1 + \lambda_2 + \lambda_3)w}}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2)} + \frac{\lambda_3 e^{- (\lambda_2 + \lambda_3)w}}{\lambda_2 + \lambda_3} - \frac{\lambda_1 e^{- \lambda_3 w}}{\lambda_1 + \lambda_2} + \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_2 + \lambda_3)}. \]
And, of course, as a special case of the above, we have
\[ \Pr(X_1 < X_2 < X_3) = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_2 + \lambda_3)}. \]

Now let \( W = \max_i X_i \). Using the above two results, we can calculate the conditional c.d.f. of \( W|X_1 < X_2 < X_3 \) by plugging into
\[ F_{W|X_1<X_2<X_3}(w) = \frac{\Pr(W \leq w|X_1 < X_2 < X_3)}{\Pr(X_1 < X_2 < X_3)}. \]
\[ = \frac{\Pr(W \leq w|X_1 < X_2 < X_3)}{\Pr(X_1 < X_2 < X_3)}. \]
Then we have the conditional p.d.f.
\[ f_{W|X_1<X_2<X_3}(w) = \frac{d}{dw} F_{W|X_1<X_2<X_3}(w), \]
and expected value

\[
E[W | X_1 < X_2 < X_3] = \int_0^\infty f_{W | X_1 < X_2 < X_3}(w) \, dw = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3},
\]

after all of the algebraic smoke clears. \(\square\)

By the way, could you have gotten this result by inspection?

(d) Summing over all 6 permutations of 1,2,3, and using part (c), we have

\[
E[\max(X_i)] = \sum_{i \neq j \neq k} E[\max(X_i) | X_i < X_j < X_k] P(X_i < X_j < X_k)
= \sum_{i \neq j \neq k} \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k} \left[ \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k} \right]. \square
\]

8. Let \(X_1\) and \(X_2\) be i.i.d. \(\text{Exp}(\mu)\). Define \(X_{(1)} = \min(X_1, X_2)\) and \(X_{(2)} = \max(X_1, X_2)\). Find

(a) \(E[X_{(1)}]\).
(b) \(\text{Var}(X_{(1)})\).
(c) \(E[X_{(2)}]\).
(d) \(\text{Var}(X_{(2)})\).

**Solution:** (a) Let’s find the distribution of \(Y \equiv X_{(1)}\). The c.d.f. is

\[
G(y) = \Pr(Y \leq y)
= 1 - \Pr(Y > y)
= 1 - \Pr(\min(X_1, X_2) > y)
= 1 - \Pr(X_1 > y \text{ and } X_2 > y)
= 1 - \Pr(X_1 > y) \Pr(X_2 > y) \quad \text{(by independence)}
= 1 - [\Pr(X_1 > y)]^2 \quad (X_1 \text{ and } X_2 \text{ are i.i.d.})
= 1 - [e^{-\mu y}]^2 \quad (\text{since } X_1 \sim \text{Exp}(\mu))
= 1 - e^{-2\mu y}
\]

Thus, we’ve shown that \(Y \sim \text{Exp}(2\mu)\). This implies that \(E[X_{(1)}] = 1/(2\mu)\). \(\square\)

(b) Similarly, \(\text{Var}(X_{(1)}) = 1/(2\mu)^2\). \(\square\)
(c) There are lots of ways to do this. One is simply to derive the p.d.f. of $X_{(2)}$ and then do the usual calculus. But I’ll show you here a method that I stole from Ross’s answer book, which is a nice, intuitive technique.

Define $A = X_{(2)} - X_{(1)}$. By the memoryless property, $A \sim \text{Exp}(\mu)$ and is independent of $X_{(1)}$. Thus,

$$E[X_{(2)}] = E[X_{(1)} + A] = \frac{1}{2\mu} + \frac{1}{\mu} = \frac{3}{2\mu}. \quad \square$$

(d) Similarly,

$$\text{Var}(X_{(2)}) = \text{Var}(X_{(1)} + A) = \frac{1}{4\mu^2} + \frac{1}{\mu^2} = \frac{5}{4\mu^2}. \quad \square$$

9. If you have nothing better to do, repeat Problem 8, but now suppose that the $X_i$’s are independent exponentials with respective rates $\mu_i, i = 1, 2$.

**Solution:** (a) and (b) The c.d.f. of $Y \equiv \min(X_1, X_2)$ is

$$\Pr(Y \leq y) = \Pr(\min(X_1, X_2) \leq y)$$

$$= 1 - \Pr(\min(X_1, X_2) > y)$$

$$= 1 - \Pr(X_1 > y, X_2 > y)$$

$$= 1 - \Pr(X_1 > y) \Pr(X_2 > y)$$

$$= 1 - e^{-(\mu_1+\mu_2)y},$$

implying that $Y \sim \text{Exp}(\mu_1 + \mu_2)$. This immediately implies that $E[Y] = 1/(\mu_1 + \mu_2)$ and $\text{Var}(Y) = 1/(\mu_1 + \mu_2)^2$. \quad \square

(c) and (d) The c.d.f. of $W \equiv \max(X_1, X_2)$ is

$$\Pr(W \leq w) = \Pr(\max(X_1, X_2) \leq w)$$

$$= \Pr(X_1 \leq w, X_2 \leq w)$$

$$= \Pr(X_1 \leq w) \Pr(X_2 \leq w)$$

$$= (1 - e^{-\mu_1y})(1 - e^{-\mu_2y})$$

$$= 1 - e^{-\mu_1y} - e^{-\mu_2y} + e^{-(\mu_1+\mu_2)y}.$$

Thus, the p.d.f. is

$$f_W(w) = \mu_1e^{-\mu_1y} + \mu_2e^{-\mu_2y} - (\mu_1 + \mu_2)e^{-(\mu_1+\mu_2)y}, \quad y \geq 0.$$
After a little algebra, we get

\[ E[W] = \int_{0}^{\infty} w f_W(w) \, dw = \frac{\mu_1^2 + \mu_1 \mu_2 + \mu_2^2}{\mu_1 \mu_2 (\mu_1 + \mu_2)} \]

and

\[ E[W^2] = \int_{0}^{\infty} w^2 f_W(w) \, dw = \frac{2(\mu_1^4 + 2\mu_1^3 \mu_2 + \mu_1^2 \mu_2^2 + 2\mu_1 \mu_2^3 + \mu_2^4)}{\mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^2} \]

so that

\[ \text{Var}(W) = E[W^2] - (E[W])^2 = \frac{\mu_1^4 + 2\mu_1^3 \mu_2 - \mu_1^2 \mu_2^2 + 2\mu_1 \mu_2^3 + \mu_2^4}{\mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^2}. \]

10. Events occur according to a Poisson process with rate \( \lambda = 2/\text{hr.} \)

(a) What’s the probability that no event occurs between 8 p.m. and 9 p.m.?

(b) Starting at noon, what’s the expected time until the fourth event occurs?

(c) What’s the probability that two or more events occur between 6 p.m. and 8 p.m.?

Solution: (a) The number of arrivals between 8PM and 9PM is Pois(2). This implies

\[ \Pr(\text{no arrivals}) = \frac{e^{-220}}{0!} = e^{-2}. \]

(b) 2PM. \( \square \)

(c) \( 1 - 5e^{-4} \). \( \square \)

11. Events occur according to a nonhomogeneous Poisson process with mean value function \( m(t) = t^2 + 2t, t \geq 0 \). What’s the probability that \( n \) events occur between times \( t = 4 \) and \( t = 5 \)?

Solution: \( N(5) - N(4) \sim \text{Pois}(m(5) - m(4)) = \text{Pois}(11) \). This implies that

\[ \Pr(N(5) - N(4) = n) = \frac{e^{-11}11^n}{n!}. \]

\( \square \)
12. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state $i$, $i = 0, 1, 2, 3$, if the first urn contains $i$ white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let $X_n$ denote the state of the system after the $n$th step. Explain why $\{X_n, n \geq 0\}$ is a Markov chain and calculate its probability transition matrix.

Solution:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad \Box$$

13. Consider the Markov chain $\{X_n, n \geq 0\}$ with states 0,1,2 whose transition probability matrix is

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $f(0) = 0$ and $f(1) = f(2) = 1$. If $Y_n = f(X_n)$, is $\{Y_n, n \geq 0\}$ a Markov chain?

Solution: Let's calculate

$$P(Y_n = 0|Y_{n-1} = 1, Y_{n-2} = 0)$$

$$= P(f(X_2) = 0|f(X_1) = 1, f(X_0) = 0)$$

$$= P(X_2 = 0|X_1 = 1 \text{ or } 2, X_0 = 0)$$

$$= P_{01}P_{10} + P_{02}P_{20}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(1\right) = \frac{3}{4}.$$

Meanwhile,

$$P(Y_n = 0|Y_{n-1} = 1, Y_{n-2} = 1, Y_{n-3} = 0)$$

$$= P(f(X_3) = 0|f(X_2) = 1, f(X_1) = 1, f(X_0) = 0)$$

$$= P(X_3 = 0|X_2 = 1 \text{ or } 2, X_1 = 1 \text{ or } 2, X_0 = 0)$$

$$= P_{01}P_{11}P_{10} + P_{01}P_{12}P_{20} + P_{02}P_{21}P_{10} + P_{02}P_{22}P_{20}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(0\right)\left(1\right) + \left(\frac{1}{2}\right)\left(0\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(0\right)\left(1\right) = \frac{1}{8}. $$
From the above two calculations, we see that the probability of \( Y_n = 0 \) depends not only on \( Y_{n-1} \) (good) but on \( Y_{n-2} \) (bad). Therefore, the \( Y_n \)'s do not form a Markov chain. □

14. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of \( H \). If the coin flipped today comes up \( H \), then we select coin 1 to flip tomorrow; and if it comes up \( T \), then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what’s the probability that the coin flipped on the third day after the initial flip is coin 1?

**Solution:**

\[
P^{(3)} = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}^3 = \begin{pmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{pmatrix}.
\]

You want \((P^{(3)}_{11} + P^{(3)}_{21})/2 = 0.6665\). □

15. Specify the classes of the following MC’s, and determine whether they are transient or recurrent.

(a)

\[
P_1 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.
\]

(b)

\[
P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

(c)

\[
P_3 = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.
\]
Solution: \( P_1 : \{0, 1, 2\} \) recurrent. \( \square \)

\( P_2 : \{0, 1, 2, 3\} \) recurrent (since \( 0 \to 3 \to 2 \to 1 \to 3 \to 2 \to 0 \)). \( \square \)

\( P_3 : \{0, 2\} \) recurrent; \( \{1\} \) transient; \( \{3, 4\} \) recurrent; \( \square \)

\( P_4 : \{0, 1\} \) recurrent; \( \{2\} \) recurrent; \( \{3\} \) transient; \( \{4\} \) transient; \( \square \)

16. A particle moves on a circle through points that have been marked 0,1,2,3,4 (in a clockwise order). At each step, it has probability \( p \) of moving to the right (clockwise) and \( 1 - p \) to the left (counterclockwise). Let \( X_n \) denote its location on the circle after the \( n \)th step. The process \( \{X_n, n \geq 0\} \) is a MC.

(a) Find the transition probability matrix.

(b) Calculate the limiting probabilities.

Solution: (a)

\[
P = \begin{pmatrix}
0 & p & 0 & 0 & 1 - p \\
1 - p & 0 & p & 0 & 0 \\
0 & 1 - p & 0 & p & 0 \\
0 & 0 & 1 - p & 0 & p \\
p & 0 & 0 & 1 - p & 0
\end{pmatrix}.
\]

(b) Since the columns of \( P \) each sum to 1, the matrix is doubly stochastic. So from class notes we know that \( \pi_j = 1/5 \) for all \( j \). \( \square \)

17. An organization has \( N \) employees, where \( N \) is a large number. Each employee has one of three possible job classifications, and changes classifications according to a
MC with transition probabilities

\[
\begin{pmatrix}
0.7 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.2 \\
0.1 & 0.4 & 0.5
\end{pmatrix}.
\]

What percentage of employees are in each classification?

**Solution:** Must solve

\[
\begin{align*}
\pi_0 &= 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2 \\
\pi_1 &= 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 \\
\pi_2 &= 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2
\end{align*}
\]

and \(\pi_0 + \pi_1 + \pi_2 = 1\). After some algebra, we eventually get

\[
\pi = \left( \frac{6}{17}, \frac{7}{17}, \frac{4}{17} \right). \quad \Box
\]

18. A taxi driver provides service in two zones of a city. Fares picked up in zone A will have destinations in zone A w.p. 0.6 or in zone B w.p. 0.4. Fares picked up in B will have destinations in A w.p. 0.3 or in B w.p. 0.7. The driver’s expected profit for a trip entirely in zone A is $6; for a trip entirely in B it’s $8; and for a trip involving both zones it’s $12. Find the driver’s average profit per trip.

**Solution:** Let the states be the successive pickup zones. Then we have \(P_{AA} = 0.6\), \(P_{BA} = 0.3\). Therefore, the long-run proportions of pickups from each zone can be calculated from

\[
\pi_A = 0.6\pi_A + 0.3\pi_B = 0.6\pi_A + 0.3(1 - \pi_A).
\]

This immediately yields \(\pi_A = 3/7\) and \(\pi_B = 4/7\).

Now let \(X\) be the profit from a trip. Conditioning on the zone gives

\[
\mathbb{E}[X] = \frac{3}{7} \mathbb{E}[X|A] + \frac{4}{7} \mathbb{E}[X|B]
\]

\[
\mathbb{E}[X] = \frac{3}{7} \left[ 0.6(6) + 0.4(12) \right] + \frac{4}{7} \left[ 0.3(12) + 0.7(8) \right] = 62/7. \quad \Box
\]
19. In the gambler’s ruin problem, let $M_i$ denote the mean number of games that must be played until the gambler either goes broke or hits $N$, given that he starts with $i$, $i = 0, 1, \ldots, N$. It can be shown (you don’t have to do this) that $M_i$ satisfies

$$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, 2, \ldots, N - 1.$$ 

Show that the solution to the above equations is

$$M_i = \begin{cases} 
  i(N - i) & \text{if } p = 1/2 \\
  \frac{i}{q-p} - \frac{N}{q-p} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 
\end{cases}.$$ 

**Solution:** You should really solve this via difference equations. Nevertheless, you can cheat (like I did) and merely verify via direct substitution that $M_i = 1 + qM_{i-1} + pM_{i+1}$ is true. $\square$