1. Suppose that the random variable $X$ has p.d.f.

$$f_X(x) = \begin{cases} \frac{20}{3}(x^{3/2} - x^3) & \text{if } 0 < x < 1 \\
0 & \text{otherwise} \end{cases}$$

and that the conditional p.d.f. of $Y$ given $X = x$ is

$$f(y|x) = \begin{cases} \frac{x^4}{4(x^{3/2} - x^3)} & \text{if } 4x^2 < y < 4\sqrt{x} \\
0 & \text{otherwise} \end{cases}.$$ 

Helpful Fact: $4x^2 < y < 4\sqrt{x}$ if and only if $\frac{y^2}{16} < x < \frac{y}{2}$.

(a) Find $E[Y|X = x]$.

**Solution:**

$$E[Y|X] = \int_{\mathbb{R}} y f(y|x) \, dy = \int_{4x^2}^{4\sqrt{x}} \frac{yx}{4(x^{3/2} - x^3)} \, dy = \frac{2(x^2 - x^5)}{x^{3/2} - x^3}.$$  

(b) Use (a) to find $E[Y]$.

**Solution:** By double expectation, we have

$$E[Y] = E[E[Y|X]] = \int_{\mathbb{R}} E[Y|X] f_X(x) \, dx = \int_0^1 \frac{2(x^2 - x^5)}{x^{3/2} - x^3} \cdot \frac{20}{3}(x^{3/2} - x^3) \, dx = \frac{20/9}{3} \int_0^1 (x^2 - x^5) \, dx = \frac{20}{27}.$$  

(c) Find the joint p.d.f. of $X$ and $Y$. 


Solution:

\[ f(x, y) = f(y|x)f_X(x) \]
\[ = \frac{x}{4(x^{3/2} - x^3)} \cdot \frac{20}{3} (x^{3/2} - x^3) \]
\[ = \frac{5x}{3}, \quad \text{for } 0 < x < 1 \text{ and } 4x^2 < y < 4\sqrt{x}. \]

(d) Use (c) to find the p.d.f. of \( Y \), i.e., \( f_Y(y) \).

Solution: Using the Helpful Fact from the statement of the problem, we have

\[ f_Y(y) = \int_{\mathbb{R}} f(x, y) \, dx \]
\[ = \int_{\sqrt{y}/2}^{\sqrt{y}/2} \frac{5x}{3} \, dx \]
\[ = \frac{5}{24} \left( y - \frac{y^4}{64} \right), \quad 0 < y < 4. \]

(e) Use (d) to find \( E[Y] \).

Solution: Using the usual definition, we have

\[ E[Y] = \int_{\mathbb{R}} y f_Y(y) \, dy \]
\[ = \int_{0}^{4} \frac{5}{24} \left( y^2 - \frac{y^5}{64} \right) \, dy \]
\[ = \frac{20}{9}. \]

2. Moment Generating Functions.

(a) Suppose that \( X \sim \text{Nor}(0, 1) \). Find the m.g.f. of \( Z = X^2 \).

Helpful Fact: \( \int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\pi/a} \).

Solution:

\[ M_Z(t) = E[e^{tZ}] \]
\[ = E[e^{tX^2}] \]
\[ = \int_{\mathbb{R}} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \]
\[
\begin{align*}
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left[ - \left( \frac{1}{2} - t \right) x^2 \right] dx \quad \text{(if } t < 1/2) \\
&= \frac{1}{\sqrt{2\pi \sqrt{1 - t}}} \quad \text{(by the Helpful Fact)} \\
&= \frac{1}{\sqrt{1 - 2t}} \quad \text{if } t < 1/2. \quad \Box
\end{align*}
\]

(b) Suppose that \( X_1, X_2, \ldots, X_n \sim \text{Nor}(0, 1) \). Find the m.g.f. of \( Y = \sum_{i=1}^{n} X_i^2 \).

**Solution:** By (a) and the fact that \( Y \) is the sum of i.i.d. \( X_i \)'s, we have

\[
M_Y(t) = \mathbb{E}[e^{tY}] = (\mathbb{E}[e^{tX_1}])^n = (1 - 2t)^{-n/2}. \quad \Box
\]

(c) Find \( \mathbb{E}[Y] \).

**Solution:**

\[
\begin{align*}
\mathbb{E}[Y] &= \left. \frac{d}{dt} M_Y(t) \right|_{t=0} \\
&= \left. \frac{d}{dt} (1 - 2t)^{-n/2} \right|_{t=0} \\
&= n(1 - 2t)^{-\frac{n}{2} - 1} \bigg|_{t=0} \\
&= n. \quad \Box
\end{align*}
\]

(d) Find \( \text{Var}(Y) \).

**Solution:** First of all, by part (c), we have

\[
\begin{align*}
\mathbb{E}[Y^2] &= \left. \frac{d^2}{dt^2} M_Y(t) \right|_{t=0} \\
&= \left. \frac{d}{dt} n(1 - 2t)^{-\frac{n}{2} - 1} \right|_{t=0} \\
&= (n + 2)n(1 - 2t)^{-\frac{n}{2} - 2} \bigg|_{t=0} \\
&= (n + 2)n.
\end{align*}
\]

This implies \( \text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = (n + 2)n - n^2 = 2n. \quad \Box \)
(e) Find $E[Y^k]$. (Don’t simplify the resulting polynomial.)

**Solution:** Using a similar argument to that employed in part (d) above, we find that $E[Y^3] = (n + 4)(n + 2)n$, and we notice a pattern. In fact,

$$E[Y^k] = (n + 2(k - 1)) \cdots (n + 4)(n + 2)n. \quad \Box$$

3. Suppose that the number of accidents per week at an industrial plant is Pois(12). Further, suppose that the numbers of workers injured in each accident are i.i.d. Bin(10, 0.1) random variables. Assuming that the number of workers injured in each accident is independent of the number of accidents that occur, find the expected value and variance of the number of injuries during a given week.

**Solution:** Let $N \sim$ Pois(12) denote the number of accidents, and suppose $X_i \sim$ Bin(10, 0.1) is the number of workers injured in accident $i$, $i = 1, 2, \ldots, N$. Denote the total number of injuries by $T \equiv \sum_{i=1}^{N} X_i$. Then, under the conditions of the problem, we have

$$E[T] = E[N]E[X_1] = 12 \cdot 10(0.1) = 12. \quad \Box$$

Further,

$$\text{Var}(T) = E[N]\text{Var}(X_1) + (E[X_1])^2\text{Var}(N)$$

$$= 12 \cdot 10(0.1)(0.9) + 1^2 \cdot 12$$

$$= 22.8. \quad \Box$$

4. Every time Shaquille O’Neal shoots the basketball, the probability that he scores is $p$. Suppose that the number of shots he takes in a particular game is Pois($\lambda$) distributed. Find the probability that he doesn’t score a basket during a particular game. Hint: Use a conditioning argument and the fact that $\sum_{n=0}^{\infty} (a^n/n!) = e^a$.

**Solution:** Let $N \sim$ Pois($\lambda$) denote the number of shots that Shaq takes, and suppose

$$X_i \sim \text{Bern}(p) = \begin{cases} 1 & \text{if he scores on shot } i \\ 0 & \text{if he misses on shot } i \end{cases}, \quad i = 1, 2, \ldots, N.$$

Denote the total number of baskets by $T \equiv \sum_{i=1}^{N} X_i$. Using a standard conditioning argument, we have

$$\Pr(T = 0) = \sum_{n=0}^{\infty} \Pr(T = 0|N = n) \Pr(N = n)$$
5. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. Pois($\lambda$).

(a) Find the m.g.f. of $X_i$.

Solution: By the Law of the Unconscious Statistician,

$$M_X(t) = E[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} \frac{e^{-\lambda} \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^{t}-1)}. \quad \square$$

(b) Use this m.g.f. to find the distribution of $\sum_{i=1}^{n} X_i$.

Solution: Let $Y_n = \sum_{i=1}^{n} X_i$. By (a) and the fact that the $X_i$’s are i.i.d., we have

$$M_{Y_n}(t) = [M_X(t)]^n = e^{n\lambda(e^t-1)}.$$
This is the m.g.f. of a Pois($n\lambda$) distribution. So by the uniqueness of m.g.f.’s (at least in this class), we have $Y_n \sim \text{Pois}(n\lambda)$.  

(c) Suppose $\lambda = 1$. Use the Central Limit Theorem to find the approximate value of $\Pr \left( 95 \leq \sum_{i=1}^{100} X_i \leq 105 \right)$.

**Solution:**

\[
\Pr(95 \leq Y_{100} \leq 105) = \Pr \left( \frac{95 - n\lambda}{\sqrt{n\lambda}} \leq \frac{Y_n - n\lambda}{\sqrt{n\lambda}} \leq \frac{105 - n\lambda}{\sqrt{n\lambda}} \right) \\
= \Pr \left( \frac{95 - 100}{10} \leq \frac{Y_{100} - 100}{10} \leq \frac{105 - 100}{10} \right) \\
\approx \Pr (-0.5 \leq \text{N}(0, 1) \leq 0.5) \quad \text{(by the CLT)} \\
= 2\Phi(0.5) - 1 = 2(0.6915) - 1 = 0.383. \quad \square
\]

6. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third leads immediately to freedom.

(a) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?

**Solution:** Let $X$ denote the door chosen, and $N$ the days spent in jail. Using the usual conditioning argument, we have

\[
\mathbb{E}[N] = \sum_{k=1}^{3} \mathbb{E}[N|X = k] \Pr(X = k) \\
= (2 + \mathbb{E}[N])(0.5) + (3 + \mathbb{E}[N])(0.3) + (0)(0.2) \\
= 1.9 + 0.8\mathbb{E}[N]
\]

This implies that $\mathbb{E}[N] = 9.5$.  \quad \square

(b) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)
**Solution:** Now let $N_i$ denote the number of additional days in jail after initially choosing door $i$. Then

$$E[N] = \frac{1}{3} \left(2 + E[N_1]\right) + \frac{1}{3} \left(3 + E[N_2]\right) + \frac{1}{3} (0) = \frac{5}{3} + \frac{1}{3} (E[N_1] + E[N_2]).$$

Meanwhile,

$$E[N_1] = \frac{1}{2} (3) + \frac{1}{2} (0) = \frac{3}{2}$$

and

$$E[N_2] = \frac{1}{2} (3) + \frac{1}{2} (0) = 1.$$

These facts imply that $E[N] = 5/2$.  

\[\Box\]