1. Suppose that the random variable $X$ has p.d.f. 
\[ f_X(x) = \begin{cases} 
\frac{20}{3}(x^{3/2} - x^3) & \text{if } 0 < x < 1 \\
0 & \text{otherwise} 
\end{cases} \]
and that the conditional p.d.f. of $Y$ given $X = x$ is 
\[ f(y|x) = \begin{cases} 
x^4(x^{3/2} - x^3) & \text{if } 4x^2 < y < 4\sqrt{x} \\
0 & \text{otherwise} 
\end{cases} \]

Helpful Fact: $4x^2 < y < 4\sqrt{x}$ if and only if $\frac{y^2}{16} < x < \frac{x}{2}$.

(a) Find $E[Y|X = x]$.
(b) Use (a) to find $E[Y]$.
(c) Find the joint p.d.f. of $X$ and $Y$.
(d) Use (c) to find the p.d.f. of $Y$, i.e., $f_Y(y)$.
(e) Use (d) to find $E[Y]$.

2. Moment Generating Functions.

(a) Suppose that $X \sim \text{Nor}(0,1)$. Find the m.g.f. of $Z = X^2$.

Helpful Fact: $\int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\pi/a}$.

(b) Suppose that $X_1, X_2, \ldots, X_n \sim \text{Nor}(0,1)$. Find the m.g.f. of $Y = \sum_{i=1}^n X_i^2$.
(c) Find $E[Y]$.
(d) Find $\text{Var}(Y)$.
(e) Find $E[Y^k]$. (Don’t simplify the resulting polynomial.)

3. Suppose that the number of accidents per week at an industrial plant is Pois(12).
Further, suppose that the numbers of workers injured in each accident are i.i.d. Bin(10, 0.1) random variables. Assuming that the number of workers injured in each accident is independent of the number of accidents that occur, find the expected value and variance of the number of injuries during a given week.

4. Every time Shaquille O’Neal shoots the basketball, the probability that he scores is $p$. Suppose that the number of shots he takes in a particular game is Pois($\lambda$) distributed. Find the probability that he doesn’t score a basket during a particular game. Hint: Use a conditioning argument and the fact that $\sum_{n=0}^{\infty} (a^n/n!) = e^a$. 

5. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. Pois($\lambda$).

   (a) Find the m.g.f. of $X_i$.
   (b) Use this m.g.f. to find the distribution of $\sum_{i=1}^{n} X_i$.
   (c) Suppose $\lambda = 1$. Use the Central Limit Theorem to find the approximate value of $\Pr \left( 95 \leq \sum_{i=1}^{100} X_i \leq 105 \right)$.

6. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third leads immediately to freedom.

   (a) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?
   (b) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)