1. An electronic assembly consists of two subsystems, say A and B. Suppose we have the following information:

- \( \Pr(\text{B fails}) = 0.5 \)
- \( \Pr(\text{A and B fail}) = 0.3 \)
- \( \Pr(\text{A fails but B doesn’t fail}) = 0.3 \)

Find the probability that B fails given that A fails.

**Solution:** We have \( \Pr(B) = 0.5, \Pr(A \cap B) = 0.3, \) and \( \Pr(A \cap \bar{B}) = 0.3 \). Thus, by the Law of Total Probability,

\[
\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A \cap B)}{\Pr(A \cap B) + \Pr(A \cap \bar{B})} = \frac{1}{2}. \quad \square
\]

2. Consider the continuous random variable \( Y \) having p.d.f.

\[
f(y) = \begin{cases} 
c|y|^3 & \text{if } -1 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Find \( c \).

**Solution:** By symmetry,

\[
1 = \int \limits_{\mathbb{R}} f(y) \, dy = 2 \int_{0}^{1} cy^3 \, dy = \frac{c}{2},
\]

so that \( c = 2 \). \quad \square

(b) Find \( \Pr(-1 \leq Y \leq 0) \).

**Solution:** By symmetry,

\[
\Pr(-1 \leq Y \leq 0) = \int_{-1}^{0} f(y) \, dy = \int_{0}^{1} 2y^3 \, dy = \frac{1}{2}. \quad \square
\]
(c) Find $\Pr(0 \leq Y \leq 0.5|0 \leq Y \leq 1)$.

**Solution:**

$$\Pr(0 \leq Y \leq 0.5|0 \leq Y \leq 1) = \frac{\Pr(0 \leq Y \leq 0.5)}{\Pr(0 \leq Y \leq 1)}$$

$$= \frac{\int_0^{1/2} 2y^3 \, dy}{\int_0^1 2y^3 \, dy}$$

$$= \frac{1}{16}.$$

(d) Find $\Pr(0 \leq Y \leq 0.5 | -1 \leq Y \leq -0.5)$.

**Solution:** 0. □

(e) Find $E[Y]$.

**Solution:**

$$E[Y] = \int_{\mathbb{R}} yf(y) \, dy = \int_{-1}^{1} 2y^4 \, dy = \frac{4}{5}.$$ □

(f) Find $\text{Var}(Y)$.

**Solution:** First of all,

$$E[Y^2] = \int_{\mathbb{R}} y^2f(y) \, dy = \int_{-1}^{1} 2|y|^3y^2 \, dy = 4 \int_{0}^{1} y^5 \, dy = \frac{2}{3}.$$

Then we have

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}.$$ □

(g) Find $E[3Y - 2]$.


(h) Find $\text{Var}(3Y - 2)$.

**Solution:** $\text{Var}(3Y - 2) = 9\text{Var}(Y) = 6/25$. □
3. TRUE-FALSE Questions. $X$ and $Y$ must be independent if

(a) $f(x|y) = f_Y(y)$ for all $y$.

Solution: FALSE. □

(b) $\text{Cov}(X, Y) = 0$.

Solution: FALSE (only converse is true). □

(c) $f(x, y) = cy$, $0 < x < y < 1$.

Solution: FALSE (funny limits). □

(d) $f(x, y) = cy^2/(1 + x^3)$, $0 < x < 1$, $1 < y < 3$.

Solution: TRUE ($f(x, y)$ factors into $g(x)h(y)$). □

(e) $E(XY) = E(X) \cdot E(Y)$.

Solution: FALSE (same as $\text{Cov} = 0$). □

4. Suppose $f(x, y) = cx$, $0 < y < x < 1$.

(a) Find $c$.

Solution: We have

$$1 = \int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_0^1 \int_0^x c \, dx \, dy = \frac{c}{3},$$

so that $c = 3$. □

(b) Find $\text{Pr}(X < 0.5 \text{ and } Y > 0.5)$.

Solution: 0. □
(c) Find the p.d.f. of $Y$.

**Solution:**

$$f_Y(y) = \int_\mathbb{R} f(x, y) \, dx = \int_y^1 3x \, dx = \frac{3}{2}(1 - y^2), \quad 0 < y < 1. \quad \Box$$

(d) Find the conditional p.d.f. of $X$ given that $Y = y$.

**Solution:**

$$f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{3x}{\frac{3}{2}(1 - y^2)} = \frac{2x}{1 - y^2}, \quad 0 < y < x < 1. \quad \Box$$

(e) Find $E[X|Y = y]$.

**Solution:**

$$E[X|y] = \int_\mathbb{R} x f(x|y) \, dx = \int_y^1 \frac{2x^2}{1 - y^2} \, dx = \frac{2}{3} \frac{(y^2 + y + 1)}{y + 1}. \quad \Box$$

5. Suppose that $E[X] = 3$, $E[Y] = 2$, $\text{Var}(X) = 5$, $\text{Var}(Y) = 4$, and $\text{Cov}(X, Y) = -2$.

(a) Find $E[2X + 3Y]$.

**Solution:** $E[2X + 3Y] = 2E[X] + 3E[Y] = 2(3) + 3(2) = 12. \quad \Box$

(b) Find $\text{Var}(2X + 3Y)$.

**Solution:** $\text{Var}(2X + 3Y) = 4\text{Var}(X) + 9\text{Var}(Y) + 12\text{Cov}(X, Y) = 32. \quad \Box$

6. If the m.g.f. of $X$ is $M_X(t) = e^{2t^2}$, find $E[X]$.

**Solution:**

$$E[X] = \left. \frac{d}{dt} M_X(t) \right|_{t=0} = \left. \frac{d}{dt} e^{2t^2} \right|_{t=0} = 4te^{2t^2} \bigg|_{t=0} = 0. \quad \Box$$
7. Suppose that a light bulb has a lifetime that is exponentially distributed with a mean of 1000 hours. Suppose the bulb has already survived 3000 hours. What’s the probability that it will survive another 1000 hours?

**Solution:** Denote the lifetime by $X \sim \text{Exp}(1/1000)$. By the memoryless property,

\[
\Pr(X > 4000 | X > 3000) = \Pr(X > 1000) = e^{-\lambda t} = e^{-1} = 0.368. \quad \square
\]