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ISyE 6739 — Test #3 Solutions

Summer 2002

This test is open notes, open books. You have *exactly 90 minutes*.

1. (Short answer sampling distribution questions — Just write your answer.)

   (a) Find $z_{0.025}$.

   ANSWER: 1.96.

   (b) Find $\Phi(1.85)$.

   ANSWER: 0.9678.

   (c) If $X \sim N(2,4)$, find $\mathsf{Pr}\{X < 4\}$.

   ANSWER: $\Pr\left(\frac{X-2}{2} < \frac{4-2}{2}\right) = \Phi(1) = 0.8413$.

   (d) Find $\chi^2_{0.025,15}$.

   ANSWER: 27.49.

   (e) If $X \sim \chi^2(3)$, find $\mathsf{Pr}\{X > 7.815\}$.

   ANSWER: 0.05.

   (f) Find $t_{0.05,15}$.

   ANSWER: 1.753.

(g) If $T \sim t(7)$, find $\mathsf{Pr}\{T < 2.365\}$.

ANSWER: 0.975.

(h) Find $F_{0.025,3,4}$.

ANSWER: 9.98.

(i) Find $F_{0.975,4,3}$.

ANSWER: $1/9.98 = 0.1002$.

(j) If $F \sim F(5,3)$, find $\mathsf{Pr}\{F < 14.88\}$.

ANSWER: 0.975.

(k) BONUS: Where are the Cucumbers from?

ANSWER: Hoboken, NJ.

2. (Short answer estimation questions — Just write your answer.)

   (a) Find the sample median of 7, 12, and 1.

   ANSWER: 7.

   (b) Find the sample standard deviation of 7, 12, and 1.

   ANSWER: $\sqrt{30.33} = 5.51$.

   (c) If $X_1, \ldots, X_n$ are i.i.d. Bern(p), what is the MLE of p?

   ANSWER: $\bar{X}$.

   (d) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, where μ and σ^2 are unknown, then S^2 is unbiased for σ^2.

   ANSWER: True.

   (e) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, where μ and σ^2 are unknown, then S^2 is the MLE for σ^2.

   ANSWER: False.

   (f) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. Exp(λ), then $1/\bar{X}$ is a MOM estimator for λ.

   ANSWER: True.

   (g) If $X_1, \ldots, X_n$ are i.i.d. with $\mathsf{E}[X_i] = 3$ and $\mathsf{Var}(X_i) = 10$, find $\mathsf{E}[S^2]$.

   ANSWER: 10.

   (h) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(0, \theta)$, then $\max_{1 \le i \le n}\{X_i\}$ is unbiased for θ.

   ANSWER: False.

(i) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(0, \theta)$, then $\max_{1 \le i \le n}\{X_i\}$ is the MLE for θ.

ANSWER: True.

(j) What does "MSE" mean?

ANSWER: mean-squared error.

(k) Suppose we have 2 estimators, T_1 and T_2, for some parameter θ. Further suppose that $\mathsf{Bias}(T_1) = 5$, $\mathsf{Var}(T_1) = 10$, $\mathsf{Bias}(T_2) = 0$, and $\mathsf{Var}(T_2) = 50$. Which estimator would you use?

ANSWER: $\mathsf{MSE}(T_1) = 35$, $\mathsf{MSE}(T_2) = 50$, so choose T_1.

(l) BONUS: Who is Dr. Paul Arnold?

ANSWER: He's one of the original Zombies.

3. Suppose X_1, X_2, X_3 are i.i.d. normal with mean 1 and variance 1. What is $\mathsf{Pr}\{2X_1 - X_2 - X_3 > 0\}$?

ANSWER: $2X_1 - X_2 - X_3 \sim \text{Nor}(2-1-1, 4+1+1) \sim \text{Nor}(0,6)$, so an easy symmetry argument says that the desired probability is 1/2.

4. Suppose $X_1, \ldots, X_n$ are i.i.d. with p.d.f.

$$f(x) \equiv \begin{cases} (\gamma+1)x^\gamma & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the MLE for γ.

ANSWER:

$$L(\gamma) = \prod_{i=1}^n f(x_i) = (\gamma+1)^n \Big(\prod_{i=1}^n x_i\Big)^\gamma.$$

Thus,

$$\ell\text{n}(L) = n\ell\text{n}(\gamma+1) + \gamma\,\ell\text{n}\Big(\prod_{i=1}^n x_i\Big),$$

and so

$$\frac{d}{d\gamma}\ell\text{n}(L) = \frac{n}{\gamma+1} + \ell\text{n}\Big(\prod_{i=1}^n x_i\Big) = 0.$$

Solving for γ gives

$$\hat{\gamma} = \frac{-n}{\ell\text{n}(\prod_{i=1}^n x_i)} - 1.$$

(b) What would your estimate for γ be if we observed the data $X_1 = 0.3$, $X_2 = 0.8$, and $X_3 = 0.5$?

ANSWER: Plugging the three observations into the above equation, we get $\hat{\gamma} = 0.415$.

5. Two machines fill plastic bottles with detergent. The fill volumes produced by the machines are (somehow known to be) $\text{Nor}(\mu_X, (0.15)^2)$ and $\text{Nor}(\mu_Y, (0.18)^2)$, respectively, where the units of measurement are in fluid ounces, and where μ_X and μ_Y are unknown. Two random samples of $n = 12$ bottles from machine 1 and $m = 10$ bottles from machine 2 are selected; and the respective sample means of the fill volumes are $\bar{X} = 30.87$ and $\bar{Y} = 30.68$. Construct a 90% two-sided confidence interval on the mean difference in fill volumes.

ANSWER:

$$\begin{aligned} \mu_X - \mu_Y &\in \bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \\ &= 30.87 - 30.68 \pm 1.645\sqrt{\frac{(0.15)^2}{12} + \frac{(0.18)^2}{10}} \\ &= 0.19 \pm 0.118. \end{aligned}$$

6. Suppose that $\mu \in [-1.5, 3.5]$ is a 95% confidence interval for the mean cost incurred by a certain inventory policy. Further suppose that this interval was based on 5 independent normally distributed observations from the underlying inventory system. Unfortunately, the boss has decided that she wants a 90% confidence interval. So what is it?

ANSWER: Notice that the confidence interval can also be written as $\mu \in 1.0 \pm 2.5$.

$$\mu \in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{\frac{S^2}{n}} = 1.0 \pm t_{0.025,4}\sqrt{\frac{S^2}{5}} = 1.0 \pm 2.776\sqrt{\frac{S^2}{5}}.$$

Equating

$$2.5 = 2.776\sqrt{\frac{S^2}{5}},$$

we find that $S^2 = 4.055$.

Thus, the 90% confidence interval is

$$\mu = 1.0 \pm t_{0.05,4}\sqrt{\frac{4.055}{5}} = 1.0 \pm 2.132\sqrt{\frac{4.055}{5}} = 1.0 \pm 1.92.$$

7. The yearly total snowfall figures for Siberacuse, NY, during the last 6 years are as follows:

$$100 \quad 103 \quad 88 \quad 72 \quad 98 \quad 121$$

The corresponding yearly total snowfall figures for Buffoonalo, NY, (which is down the road from Siberacuse) are:

$$90 \quad 95 \quad 72 \quad 68 \quad 95 \quad 110$$

Find a 95% confidence interval for the difference in means.

ANSWER: We'll use a *paired-t* test here, since things match up naturally.

The respective differences (D_i's) are

$$10 \quad 8 \quad 16 \quad 4 \quad 3 \quad 11$$

The required CI is

$$\begin{aligned} \mu_D &\in \bar{D} \pm t_{\alpha/2,n-1}\sqrt{\frac{S_D^2}{n}} \\ &= 8.67 \pm t_{0.025,5}\sqrt{\frac{23.07}{6}} \\ &= 8.67 \pm 2.571\sqrt{\frac{23.07}{6}} \\ &= 8.67 \pm 5.04. \end{aligned}$$