NAME →

ISyE 6739 — Test 2 Solutions — Summer 2017

This is a take-home test. But please limit the total work time to less than about 3 hours. If you have a question, please send me an email or give me a call.

1. Suppose that X has moment generating function $M_X(t) = 3e^{2t}/(3-t)$ for $t < 3$.

   (a) Find $\mathsf{E}[X]$.

   Solution: First let's do it the most straightforward way.

$$\begin{aligned}
\mathsf{E}[X] &= \frac{d}{dt} M_X(t)\Big|_{t=0} \\
&= \frac{d}{dt} \frac{3e^{2t}}{3-t}\Big|_{t=0} \\
&= \frac{(3-t)\,6e^{2t} + 3e^{2t}}{(3-t)^2}\Big|_{t=0} \\
&= \frac{21e^{2t} - 6te^{2t}}{(3-t)^2}\Big|_{t=0} \\
&= 7/3. \quad \square
\end{aligned}$$

   Here's another way to do the problem, which involves identifying the underlying distribution as a linear function of an exponential random variable. Namely, note that $M_X(t) = e^{2t} M_Y(t)$, where $Y \sim \text{Exp}(3)$, thus implying (from a HW problem, that $X = Y + 2$, so that $\mathsf{E}[X] = \frac{1}{3} + 2 = 7/3$. $\square$

   (b) Find $\mathsf{Var}(X)$. (Be patient... this gets a little tedious!)

   Solution: Of course, you can use the straightforward way and take the second derivative to get $\mathsf{E}[X^2]$ and then $\mathsf{Var}(X)$, but I'll be lazy here and use the fact that $X = Y + 2$ to obtain $\mathsf{Var}(X) = \mathsf{Var}(Y) = 1/9$. $\square$

2. Suppose X and Y are discrete random variables with the following joint p.m.f., where any letters denote probabilities that you might need to figure out.

$f(x,y)$	$X=-3$	$X=0$	$X=5$	$\mathsf{P}(Y=y)$
$Y=1.6$	a	c	0.1	0.3
$Y=27$	b	0	0.3	d
$\mathsf{P}(X=x)$	e	0.2	f	g

(a) Fill in the table for a, b, ...

Solution: Easy to show that we have

$f(x,y)$	$X=-3$	$X=0$	$X=5$	$\mathsf{P}(Y=y)$
$Y=1.6$	0	0.2	0.1	0.3
$Y=27$	0.4	0	0.3	0.7
$\mathsf{P}(X=x)$	0.4	0.2	0.4	1.0

(b) Find $\mathsf{P}(X \le 0)$.

Solution: Then $\mathsf{P}(X \le 0) = \mathsf{P}(X=-3) + \mathsf{P}(X=0) = 0.6$. □

(c) Find $\mathsf{E}[X]$.

Solution: $\mathsf{E}[X] = \sum_x x\,\mathsf{P}(X=x) = (-3)(0.4) + 0(0.2) + 5(0.4) = 0.8$. □

(d) Find $\mathsf{Cov}(X,Y)$.

Solution: Similarly, we have

$$\mathsf{E}[Y] \;=\; \sum_y y\,\mathsf{P}(Y=y) \;=\; 19.38$$

and

$$\mathsf{E}[XY] \;=\; \sum_x \sum_y xyf(x,y) \;=\; 8.9,$$

so that

$$\mathsf{Cov}(X,Y) \;=\; \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] \;=\; -6.604. \quad \Box$$

(e) Are X and Y independent?

Solution: No (since $\mathsf{Cov} \neq 0$). □

3. Suppose that $f(x, y) = cx$, for $0 \le y \le x \le 2$.

   (a) Find c.

   Solution: $1 \;=\; \int_0^2 \int_0^x cx\,dy\,dx \;=\; 8c/3$, so that $c = 3/8$. □

   (b) Find $\mathsf{P}(X > 1 \text{ and } Y < 1/2)$.

   Solution: $\int_1^2 \int_0^{1/2} \frac{3x}{8}\,dy\,dx \;=\; 9/32$. □

   (c) Find the marginal p.d.f. of X.

   Solution: $f_X(x) \;=\; \int_0^x \frac{3x}{8}\,dy \;=\; \frac{3x^2}{8},\quad 0 \le x \le 2$. □

   (d) Find the conditional p.d.f. of Y given that $X = x$.

   Solution: $f(y|x) \;=\; \frac{f(x,y)}{f_X(x)} \;=\; \frac{1}{x},\quad 0 \le y \le x$. □

   (e) Find $\mathsf{E}[Y|X = x]$.

   Solution: $\mathsf{E}[Y|x] \;=\; \int_0^x y f(y|x)\,dy \;=\; \frac{x}{2},\quad 0 \le x \le 2$. □

   (f) Find $\mathsf{E}\Big[\mathsf{E}[Y|X]\Big]$.

   Solution: $\int_0^2 \mathsf{E}[Y|x] f_X(x)\,dx \;=\; \int_0^2 \frac{x}{2}\frac{3x^2}{8}\,dx \;=\; 3/4$. □

   By the way, note that

   $$f_Y(y) \;=\; \int_y^2 \frac{3x}{8}\,dx \;=\; \frac{3(4-y^2)}{16},\quad 0 \le y \le 2.$$

   This implies that

   $$\mathsf{E}[Y] \;=\; \int_0^2 y\,\frac{3(4-y^2)}{16}\,dy \;=\; 3/4,$$

   which makes sense since $\mathsf{E}\Big[\mathsf{E}[Y|X]\Big] = \mathsf{E}[Y]$.

(g) Find $\mathsf{Cov}(X,Y)$.

Solution: First of all,

$$\mathsf{E}[X] \;=\; \int_0^2 x\,\frac{3x^2}{8}\,dx \;=\; 3/2,$$

and

$$\mathsf{E}[XY] \;=\; \int_0^2\int_0^x \frac{3x^2y}{8}\,dy\,dx \;=\; 6/5.$$

This implies that $\mathsf{Cov}(X,Y) = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] \;=\; 3/40.$ □

(h) Are X and Y independent?

Solution: No. □

4. Let's play Name That Distribution! In each question, name the distribution (with parameters, if appropriate).

(a) I am the sum of 15 i.i.d. Bern(0.3) random variables.

Solution: Bin$(15, 0.3)$. □

(b) I am the only discrete distribution with the memoryless property.

Solution: Geometric. □

(c) I am the only continuous distribution with the memoryless property.

Solution: Exponential. □

(d) I am the limiting distribution of $\sqrt{n}(\bar{X} - \mu)/\sigma$ as $n \to \infty$.

Solution: Nor(0,1). □

(e) I am the time between two arrivals from a Poisson process with rate 1/(3 hr).

Solution: Exp(1/3). □

(f) I have m.g.f. $\exp(3t + 4t^2)$.

Solution: The Nor(μ, σ^2) m.g.f. is $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$, so the answer here is Nor$(3, 8)$. $\Box$

(g) I can be represented as $F(X)$, where X is a Normal$(-2, 225)$ random variable and $F(x)$ is its p.d.f.

Solution: Unif(0,1) (by the Inverse Transform Theorem). $\Box$

5. Buses show up at the bus stop randomly according to a Poisson process with a rate of 3 per hour. Let's suppose that I arrive at the stop at 4:00PM; but, sadly, I'm still waiting for a bus at 5:00PM. What's the probability that a bus will finally arrive within the next 30 minutes?

Solution: Let X denote the waiting time. Then by the memoryless property,

$$\begin{aligned} \mathsf{P}(X \le 90 | X > 60) &= 1 - \mathsf{P}(X > 90 | X > 60) \\ &= 1 - \mathsf{P}(X > 30) \\ &= 1 - e^{-3/2} \;=\; 0.777. \quad \Box \end{aligned}$$

6. Suppose that I buy a package of 3 lightbulbs, and that their lifetimes are i.i.d. Exponential with a mean of 6 months. When a bulb fails, I immediately replace it with the next one from the package. What's the probability that the total lifetime of the bulbs will be at least a year?

Solution: The total lifetime X is Erlang$_3$(2/year). Thus,

$$\mathsf{P}(X > 1) \;=\; \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \;=\; \sum_{i=0}^{2} e^{-2} \frac{2^i}{i!} \;=\; 5e^{-2} \;=\; 0.677. \quad \Box$$

7. (This is a bit of an open-ended problem.) The *failure rate* of a positive random variable X can be regarded as the instantaneous rate of death — that is, the rate of death, given that the person (or lightbulb) has survived until time x. It's formally

defined as $f(x)/(1-F(x))$, where $f(x)$ and $F(x)$ are the p.d.f. and c.d.f. of X. Meanwhile, you may recall that the Weibull(a,b) distribution has c.d.f.

$$F(x) \; = \; 1 - \exp[-(ax)^b], \;\; x > 0.$$

Study the failure rate of $X \sim$ Weibull(a,b) for various choices of a and b.

Solution: Various solutions. You will find that for some choices of a and b, the failure rate is monotone increasing, for some it's monotone decreasing, and for $b = 1$ (the exponential case) it's flat. □

8. Median Questions.

   (a) TRUE or FALSE? The mean of any normal distribution is the same as its median. [Note that the median of a continuous RV X is simply the point x such that $\mathsf{P}(X \le x) = \mathsf{P}(X \ge x) = 0.5$.]

   Solution: TRUE (by symmetry). □

   (b) TRUE or FALSE? The mean of the Exp(λ) is greater than its median.

   Solution: By definition, we can get the median $\tilde{m}$ by solving

   $$0.5 \; = \; F(\tilde{m}) \; = \; 1 - e^{-\lambda\tilde{m}},$$

   so that

   $$\tilde{m} \; = \; \frac{\ell\text{n}(2)}{\lambda} \; = \; \frac{0.693}{\lambda} \; < \; \frac{1}{\lambda} \; = \; \mathsf{E}[X].$$

   So the answer is TRUE. □

9. Suppose $X \sim \mathsf{Nor}(1,9)$. Find the point x such that $\mathsf{P}(-x \le X \le x) = 0.95$. (Careful — this isn't quite symmetric.)

Solution: We have

$$0.95 \; = \; \mathsf{P}(-x \le X \le x) \; = \; \mathsf{P}\left(\frac{-x-1}{3} \le Z \le \frac{x-1}{3}\right) \; = \; \Phi\left(\frac{x-1}{3}\right) - \Phi\left(\frac{-x-1}{3}\right).$$

Now go to the tables, and find by trial-and-error (or via bisection) the value of x that approximately solves the equation. You'll get $x \approx 6.1923$. □

10. Suppose that a random man's height from a certain population is Nor(70,9) and a random woman's height is Nor(66,16). Find the probability that a random woman is shorter than a random man.

    Solution: Note that $W - M \sim \text{Nor}(-4, 25)$. Then we have

$$\mathsf{P}(W < M) \;=\; \mathsf{P}(W - M < 0) \;=\; \mathsf{P}\left(Z < \frac{0-(-4)}{\sqrt{25}}\right) \;=\; \Phi(0.8) \;=\; 0.788. \quad \square$$

11. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. $\mathsf{Nor}(\mu, 25)$, and let $\bar{X}$ denote the sample mean of these n observations. What's the smallest value of n that will guarantee that $\mathsf{P}\Big(|\bar{X} - \mu| \le 0.5\Big) \ge 0.995$?

    Solution: We want

$$\begin{aligned} 0.995 \;&\le\; \mathsf{P}\Big(|\bar{X} - \mu| \le 0.5\Big) \\ &=\; \mathsf{P}\Big(-0.5 \le \bar{X} - \mu \le 0.5\Big) \\ &=\; \mathsf{P}\left(\frac{-0.5}{\sqrt{25/n}} \le Z \le \frac{0.5}{\sqrt{25/n}}\right) \\ &=\; \mathsf{P}\Big(-0.1\sqrt{n} \le Z \le 0.1\sqrt{n}\Big) \\ &=\; 2\Phi(0.1\sqrt{n}) - 1. \end{aligned}$$

    This occurs if and only if $\Phi(0.1\sqrt{n}) \ge 0.995$, so that we need

$$n \;\ge\; 100\Big(\Phi^{-1}(0.9975)\Big)^2 \;=\; 100(2.807)^2 \;=\; 787.9 \;\approx\; 788. \quad \square$$

12. There are exactly 225 cookies in a bag of Biebler chocolate chip cookies. Let X_i denote the number chocolate chips in cookie i. If $X_1, \ldots, X_{225}$ are i.i.d. from a Pois(9) distribution, find the approximate probability that the total number of chocolate chips in the entire bag is at least 925.

    Solution: Oops! I messed up this question in the sense that the answer is obviously about 1. So, hopefully, you get free points!

I *meant* to ask the question for $n = 100$ cookies, in which case we'd have the following solution:

Solution to Intended Question: By the CLT,

$$\sum_{i=1}^{100} X_i \approx \text{Nor}(n\mu, n\sigma^2) \sim \text{Nor}(n\lambda, n\lambda) \sim \text{Nor}(900, 900).$$

Thus,

$$\mathsf{P}\left(\sum_{i=1}^{100} X_i > 925\right) \approx \mathsf{P}\left(Z > \frac{925 - 900}{\sqrt{900}}\right) = 1 - \Phi(5/6) = 0.202. \quad \Box$$

13. What is your favorite memory about Justin Bieber?