NAME →

ISyE 6739 — Test 2a Solutions — Summer 2009

This test is 90 minutes long. Good luck!

Put your nice, simple answers here...

1. ______ 2. ______ 3. ______ 4. ______

5. ______ 6. ______ 7. ______ 8. ______

9. ______ 10. ______ 11. ______ 12. ______

13. ______ 14. ______ 15. ______ 16. ______

17. ______ 18. ______ 19. ______ 20. ______

21. ______ 22. ______ 23. ______ 24. ______

25. ______ 26. ______ 27. ______ 28. ______

29. ______ 30. ______ 31. ______ 32. ______

33. ______ 34. ______ 35. ______ 36. ______

37. ______ 38. ______ 39. ______ 40. ______

1. Suppose that X is discrete with $f(-2) = 0.2$, $f(0) = 0.2$, and $f(4) = 0.6$. Find $F(2)$, where $F(x)$ is the c.d.f. of X.

   Solution: $F(2) = \Pr(X \le 2) = 0.4$. $\square$

2. Suppose X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. Find $\Pr(2/3 \le X \le 5/4)$.

   Solution: $\Pr(2/3 \le X \le 5/4) = \int_{2/3}^{5/4} f(x)\,dx = \int_{2/3}^{1} 2x\,dx = x^2 \mid_{2/3}^{1} = 1 - \frac{4}{9} = \frac{5}{9}$. $\square$

3. If $f(y) = 3y^2$, $0 < y < 1$, find $\mathsf{E}[1/Y^2]$.

   Solution: By the Law of the Unconscious Statistician, $\mathsf{E}[1/Y^2] = \int_R \frac{1}{y^2} f(y)\,dy = \int_0^1 \frac{1}{y^2} 3y^2\,dy = 3$. $\square$

4. TRUE or FALSE? $\mathsf{Var}(X) \ge 0$ for any random variable X.

   Solution: TRUE. $\mathsf{Var}(X) = \mathsf{E}[(X - \mu)^2] \ge 0$. $\square$

5. If $\Pr(X = 0) = 0.3$ and $\Pr(X = 1) = 0.7$, name the distribution of X (including any parameter(s)).

   Solution: $X \sim \text{Bern}(0.7)$. $\square$

6. If $\Pr(X = 0) = 0.3$ and $\Pr(X = 1) = 0.7$, find $\mathsf{E}[\ell\text{n}(X + 1)]$.

   Solution: By the Unconscious Statistician,

$$
\begin{aligned}
\mathsf{E}[\ell\text{n}(X+1)] &= \sum_x \ell\text{n}(x+1) f(x) \\
&= \ell\text{n}(0+1) f(0) + (\ell\text{n}(1+1)) f(1) \\
&= 0 + (\ell\text{n}(2))(0.7) \\
&= 0.485. \quad \square
\end{aligned}
$$

7. Suppose that the probability that GT wins any football game in 0.6, and that all games are (somehow) independent. What is the probability that GT will win exactly 3 out of its next 6 games?

   Solution: Suppose X denotes the number of games won. Then $X \sim \text{Bern}(6, 0.6)$, and we have $\Pr(X = 3) = \binom{n}{k} p^k q^{n-k} = \binom{6}{3}(0.6)^3(0.4)^3$. $\square$

8. Consider a lightbulb whose lifetime is exponential with a mean of 1 year. What's the probability that the bulb will live at least one year before failing?

   Solution: Let $X \sim \text{Exp}(1)$ denote the lifetime. Then $\Pr(X > 1) = e^{-1}$. $\square$

9. Again consider a lightbulb with an Exp(1) lifetime. Suppose that the bulb has already survived two years. What's the probability that the bulb will live at least *another* year?

   Solution: By the exponential distributions's memoryless property, $\Pr(X > 3 | X > 2) = \Pr(X > 1) = e^{-1}$. $\square$

10. If $\Pr(X = 1) = 0.4$ and $\Pr(X = 3) = 0.6$, find the moment generating function of X, i.e., $M_X(t) = \mathsf{E}[e^{tX}]$.

    Solution: $\mathsf{E}[e^{tX}] = \sum_x e^{tx} f(x) = 0.4e^t + 0.6e^{3t}$. $\square$

11. If $U \sim$ Unif(0,1), what is the distribution of $-(1/2)\ell\text{n}(1 - U)$?

    Solution: By class notes, this is Exp(2). $\square$

12. Suppose that X has p.d.f. $\frac{1}{\sqrt{8\pi}} e^{-(x-3)^2/8}$ for $-\infty < x < \infty$.What is the expected value of X?

    Solution: First, note that if X were a $N(\mu, \sigma^2)$ random variable, then it would have p.d.f. $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. We immediately see that in this problem, X is

$N(3,4)$, and thus, $\mathsf{E}[X] = 3$. $\Box$

13. What distribution do I get by using the Excel operation `RAND()*6`?

Solution: $U(0,6)$. $\Box$

14. The *kurtosis* of a random variable X is a measure of sharply peaked it is. If X has mean μ and variance σ^2, then the kurtosis is defined as

$$K = \frac{\mathsf{E}[(X-\mu)^4]}{\sigma^4} - 3 = \frac{\mathsf{E}[X^4] - 4\,\mathsf{E}[X^3]\mu + 6\,\mathsf{E}[X^2]\mu^2 - 4\,\mathsf{E}[X]\mu^3 + \mu^4}{\sigma^4} - 3.$$

Calculate the kurtosis of $X \sim \text{Bern}(1/2)$. **Hint:** From class notes, we know that for the $\text{Bern}(p)$ distribution, $\mathsf{E}[X^k] = p$, for $k = 1, 2, \ldots$.

Solution: Since $\mathsf{E}[X^k] = p$ for all k, we have

$$\begin{aligned} K &= \frac{\mathsf{E}[X^4] - 4\,\mathsf{E}[X^3]\mu + 6\,\mathsf{E}[X^2]\mu^2 - 4\,\mathsf{E}[X]\mu^3 + \mu^4}{\sigma^4} - 3 \\ &= \frac{p - 4p^2 + 6p^3 - 4p^4 + p^4}{\sigma^4} - 3 = -2. \quad \Box \end{aligned}$$

15. Suppose that $X_1, X_2, \ldots, X_5$ are $\text{Bern}(1/3)$ trials. What's the distribution of $\sum_{i=1}^5 X_i$? (Give name + parameter(s).)

Solution: $\text{Bin}(5, 1/3)$. $\Box$

16. TRUE or FALSE? If $\mathsf{Cov}(X,Y) = 0$ and X and Y are both continuous random variables, then X and Y are independent.

Solution: FALSE. (See counterexample in notes.) $\Box$

17. Suppose that $f(x,y) = cxy$, for $0 \le x \le y \le 1$, where c is the appropriate constant. Are X and Y independent?

Solution: No (funny limits). $\Box$

18. Suppose that X and Y are independent Exp(λ) random variables. Find $\mathsf{E}[\frac{X}{X+Y}]$.

    Solution: Note that $1 = \mathsf{E}[\frac{X+Y}{X+Y}]$. Since X and Y are i.i.d., we have $\mathsf{E}[\frac{X}{X+Y}] = \mathsf{E}[\frac{Y}{X+Y}] = \frac{1}{2}$. $\square$

19. Suppose X and Y are two random variables with $\mathsf{E}[X] = 1$, $\mathsf{Var}(X) = 4$, $\mathsf{E}[Y] = -1$, $\mathsf{Var}(Y) = 4$, and $\mathsf{Cov}(X, Y) = 1$. Find the correlation between X and Y.

    Solution:
$$\mathsf{Corr}(X, Y) = \frac{\mathsf{Cov}(X, Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} = \frac{1}{4}. \quad \square$$

20. Under the same conditions as in Problem 19, find $\mathsf{Var}(X - Y)$.

    Solution: $\mathsf{Var}(X - Y) = \mathsf{Var}(X) + \mathsf{Var}(Y) - 2\mathsf{Cov}(X, Y) = 6$. $\square$

21. Suppose that $f(x, y) = cy$ for $0 < y < x < 1$. Find c.

    Solution: $1 = \int\int_{R^2} f(x, y)\, dy\, dx = \int_0^1 \int_0^x cy\, dydx = \frac{c}{6} \Rightarrow c = 6$. $\square$

22. Again suppose that $f(x, y) = cy$ for $0 < y < x < 1$. Find $f_X(x)$.

    Solution: $f_X(x) = \int_R f(x, y)\, dy = \int_0^x 6y\, dy = 3x^2$, $0 < x < 1$. $\square$

23. Yet again suppose that $f(x, y) = cy$ for $0 < y < x < 1$. Find $\Pr(X < Y)$.

    Solution: Since $0 < y < x < 1$, it is obvious that
$$\Pr(X < Y) = \int\int_{x<y} f(x, y)\, dydx = 0. \quad \square$$

24. Still again suppose that $f(x, y) = cy$ for $0 < y < x < 1$. Find $f(y|x)$.

    Solution: $f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6y}{3x^2} = \frac{2y}{x^2}$, $0 < y < x < 1$. $\square$

25. TRUE or FALSE? For any random variables X and Y, we have $\mathsf{E}[\mathsf{E}[X^2|Y]] = \mathsf{E}[X^2]$.

    Solution: TRUE. $\square$

26. Find $z_{0.05}$.

    Solution: $z_{0.05} = 1.645$. $\square$

27. Find $\Phi^{-1}(0.6)$.

    Solution: $\Phi^{-1}(0.6) = 0.253$. $\square$

28. If $X \sim N(1, 4)$, find $\mathsf{Pr}\{X < 3\}$.

    Solution: $P(X < 3) = P(Z < \frac{3-\mu}{\sigma}) = P(Z < 1) = \Phi(1) = 0.8413$. $\square$

29. Find $\chi^2_{0.05,10}$.

    Solution: $\chi^2_{0.05,10} = 18.31$. $\square$

30. TRUE or FALSE? As its degrees of freedom goes to ∞, the $t(n)$ distribution approaches the standard normal.

    Solution: TRUE. $\square$

31. Find the sample variance of 2, 4, and 3.

    Solution: $S^2 = \frac{1}{n-1}(\sum X_i^2 - n\bar{X}^2) = 1$. $\square$

32. TRUE or FALSE? The sample mean is unbiased for the true mean.

    Solution: TRUE. $\square$

33. If $X_1, \ldots, X_n$ are i.i.d. Pois(λ), what is the MLE of λ?

Solution: By class notes (or doing it from scratch), $\hat{\lambda} = \bar{X}$. $\Box$

34. TRUE or FALSE? Suppose that $X_1, \ldots, X_n$ are i.i.d. Nor(μ, σ^2). Then the sample variance is the MLE for σ^2.

Solution: By class notes, FALSE. $\Box$

35. TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. Exp(λ), then $1/\bar{X}$ is unbiased for λ.

Solution: By class notes, FALSE. $\Box$

36. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. Exp(λ) and that $\bar{X} = 3$. What is the MLE for $\Pr(X > 6)$?

Solution: By the invariance property of MLE's, $\Pr(X > 6) = e^{-\hat{\lambda}t} = e^{-t/\bar{X}} = e^{-6/3} = e^{-2}$. $\Box$

37. Suppose that $X_1 = 11$, $X_2 = 25$, and $X_3 = 8$ i.i.d. observations from a Unif(0,θ) distribution. What is the MLE for θ?

Solution: $\hat{\theta} = \max X_i = 25$. $\Box$

38. If $[L, U]$ is a 95% confidence interval for some parameter θ, about what percentage of the time (over many experiments) will the confidence interval actually cover θ?

Solution: 95%. $\Box$

39. Suppose that $X_1, X_2, \ldots, X_{16}$ are i.i.d. normal with mean μ and known variance $\sigma^2 = 4$. If the sample mean $\bar{X} = 5$, what find a 95% confidence interval for μ.

Solution: $\mu \in \bar{X} \pm z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} = 5 \pm 1.96\sqrt{\frac{4}{16}} = 5 \pm 0.98$. $\Box$

40. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. normal with mean μ and known variance $\sigma^2 = 4$. What is the smallest number of observations that we will need to take to assure that the half-length of the corresponding two-sided 95% confidence interval for μ is less than 1?

    Solution: $n \geq \frac{z^2_{\alpha/2}\sigma^2}{\epsilon^2} = \frac{(1.96)^2(4)}{1} = 15.37$. Thus, take $n = 16$. $\Box$