ISyE 6739 — Test #2 Solutions

Summer 2005

This test is open notes, open books. You have *exactly 90 minutes.* Just do the best you can, and good luck!

1. (30 points) Short Answer Questions.

   (a) Suppose X has p.d.f. $f(x) = 5x^4$, $0 \le x \le 1$. Find $\mathsf{E}[2X - 5]$.

   ANSWER: $2\mathsf{E}[X] - 5 = -10/3$.

   (b) If X again has p.d.f. $f(x) = 5x^4$, $0 \le x \le 1$, find $\mathsf{Var}(2X - 5)$.

   ANSWER: $4\mathsf{Var}(X) = 0.0793$.

   (c) Suppose X can equal 1 or 2, each with probability 1/2. Find $\mathsf{E}[\ell\text{n}(X)]$.

   ANSWER: $\frac{1}{2}\,\ell\text{n}(2) = 0.347$.

   (d) TRUE or FALSE? $\mathsf{E}[X^2] \ge (\mathsf{E}[X])^2$.

   ANSWER: True.

   (e) Suppose X has m.g.f. $M_X(t) = 0.3e^t + 0.7$. What's the distribution of X?

   ANSWER: Bern(0.3).

   (f) If X has m.g.f. $4/(4 - t)$, for $t < 4$, find the m.g.f. of $2X - 1$.

   ANSWER: $4e^{-t}/(4 - 2t)$, $t < 2$.

   (g) TRUE or FALSE? If $\mathsf{Cov}(X, Y) = 0$, then X and Y are independent.

   ANSWER: False.

(h) Suppose that X and Y are independent Exponential(λ) random variables. What is the m.g.f. of $X + Y$?

ANSWER: $(\lambda/(\lambda - t))^2$.

(i) Suppose that X and Y are independent Exponential(λ) random variables. Find $\mathsf{Var}(XY)$ (yup — the variance of the product).

ANSWER: $\mathsf{E}[X^2Y^2] - (\mathsf{E}[XY])^2 = \mathsf{E}[X^2]\mathsf{E}[Y^2] - (\mathsf{E}[X])^2(\mathsf{E}[Y])^2 = 3/\lambda^4$.

(j) What does "i.i.d." mean?

ANSWER: independent and identically distributed.

(k) Bonus Question: If you were Dutch, what would "Q-65" mean to you?

ANSWER: They later became Golden Earring.

2. (10 points) Suppose that Joey sends valentines to X women, where $X \sim \text{Pois}(150)$. Each valentine will result in getting Joey Y_i dates, where $Y_i \sim \text{Geom}(0.3)$. Assuming that all of the random variables in this scenario are independent, find the *expected value* of the total number of dates Joey receives after sending out all of those valentines. (For your information, $\mathsf{E}[Y] = 1/0.3$ and $\mathsf{Var}(Y) = 0.7/(0.3)^2$.) **BONUS**: Which Brady Bunch family member was born on Valentine's Day?

ANSWER:

$$\mathsf{E}\left[\sum_{i=1}^{X} Y_i\right] = \mathsf{E}[X]\mathsf{E}[Y_1] = 150(1/0.3) = 500.$$

3. (10 points) Suppose X and Y are two random variables such that $\mathsf{E}[X] = 3$, $\mathsf{Var}(X) = 4$, $\mathsf{E}[Y] = -2$, $\mathsf{Var}(Y) = 9$, and $\mathsf{Cov}(X,Y) = -2$.

   (a) Find the correlation between $2X$ and $3Y$.

   ANSWER:

$$\mathsf{Corr}(2X, 3Y) = \frac{\mathsf{Cov}(2X, 3Y)}{\sqrt{\mathsf{Var}(2X)\mathsf{Var}(3Y)}} = \frac{\mathsf{Cov}(X,Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} = -1/3.$$

   (b) Find $\mathsf{Var}(X - Y)$.

   ANSWER: $\mathsf{Var}(X) + \mathsf{Var}(Y) - 2\mathsf{Cov}(X,Y) = 17$.

4. (10 points) Suppose X has p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$. What is the p.d.f. of $Y = \sqrt{X}$?

ANSWER: The c.d.f. of Y is

$$\begin{aligned} G(y) &= \Pr(Y \le y) = \Pr(\sqrt{X} \le y) \\ &= \Pr(X \le y^2) = \int_0^{y^2} 3x^2\, dx \\ &= y^6. \end{aligned}$$

Thus, the required p.d.f. is $g(y) = 6y^5$, $0 \le y \le 1$.

5. (20 points) Suppose that $f(x, y) = cxy$ if $0 < y < x < 1$.

(a) Find c.

ANSWER: Use $1 = \int_0^1 \int_0^x cxy\, dy\, dx$ to get $c = 8$.

(b) Find $f_X(x)$.

ANSWER: $\int_0^x 8xy\, dy = 4x^3$, $0 < x < 1$.

(c) Find $\mathsf{Corr}(X, Y)$.

ANSWER: Using the usual methods, we have

$$\mathsf{E}[X] = \frac{4}{5}, \quad \mathsf{Var}(X) = \frac{2}{75}, \quad \mathsf{E}[Y] = \frac{8}{15}, \quad \mathsf{Var}(Y) = \frac{11}{225},$$

and

$$\mathsf{E}[XY] = \int_0^1 \int_0^x 8x^2y^2\, dy\, dx = \frac{4}{9}.$$

Thus,

$$\rho = \frac{\mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} = 0.492.$$

(d) Find $f(y|x)$.

ANSWER: $f(x, y)/f_X(x) = 2y/x^2$, $0 < y < x < 1$.

(e) Use the above answer to find $\mathsf{E}[Y]$.

ANSWER: First,

$$\mathsf{E}[Y|x] = \int_{\Re} yf(y|x)\, dy = \int_0^x \frac{2y^2}{x^2}\, dy = \frac{2x}{3}.$$

Then

$$\mathsf{E}[Y] = \mathsf{E}[\mathsf{E}[Y|X]] = \int_{\Re} \mathsf{E}[Y|x] f_X(x)\, dx = \frac{8}{15}.$$

6. (20 points) Suppose X is a discrete random variable whose only possible values are the nonnegative integers. Define the *probability generating function* of X by

$$g_X(s) \;\equiv\; \mathsf{E}[s^X] \;=\; \sum_{k=0}^{\infty} s^k \Pr(X = k).$$

(a) Prove $\mathsf{E}[X] = \frac{d}{ds}\, g_X(s)|_{s=1}$.

ANSWER:

$$\frac{d}{ds}\, g_X(s)|_{s=1} \;=\; \frac{d}{ds}\,\mathsf{E}[s^X]\Big|_{s=1} \;=\; \mathsf{E}\Big[\frac{d}{ds} s^X\Big]\Big|_{s=1} \;=\; \mathsf{E}\Big[X s^{X-1}\Big]\Big|_{s=1} \;=\; \mathsf{E}[X].$$

(b) If $X \sim \text{Pois}(\lambda)$, find $g_X(s)$. (You may need the fact that $\sum_{k=0}^{\infty} y^k/k! = e^y$.)

ANSWER:

$$g_X(s) \;=\; \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!} \;=\; e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} \;=\; e^{\lambda(s-1)}.$$

(c) Suppose that $X \sim \text{Pois}(\lambda)$. Use the above two parts of the problem to find $\mathsf{E}[X]$.

ANSWER:

$$\mathsf{E}[X] \;=\; \frac{d}{ds}\, g_X(s)|_{s=1} \;=\; \lambda e^{\lambda(s-1)}\Big|_{s=1} \;=\; \lambda.$$