ISyE 6739 — Test #2 Solutions

Summer 2002

This test is open notes, open books. You have *exactly 90 minutes.*

1. Quickie Probability Questions — Just Write Your Answer.

   (a) If X is a continuous random variable with p.d.f. $f(x) = (1/2)e^{-x/2}$ for $x \geq 0$, find $\mathsf{E}[X]$.

   ANSWER: 2. □

   (b) Suppose that X has m.g.f. $M_X(t) = pe^t + q$. What's $\mathsf{E}[X]$?

   ANSWER: $M'_X(t) = pe^t|_{t=0} = p$. □

   (c) TRUE or FALSE? The Law of the Unconscious Statistician states that, if $g(\cdot)$ is a continuous function and X is a random variable, then $\mathsf{E}[g(X)] = g(\mathsf{E}[X])$.

   ANSWER: False. □

   (d) TRUE or FALSE? If X and Y are independent continuous RV's, then

   $$\Pr(X \leq Y) \;=\; \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(x,y)\, dx\, dy.$$

   ANSWER: True. □

   (e) TRUE or FALSE? $\mathsf{E}[\mathsf{E}(X|Y)] = \mathsf{E}[X]$.

   ANSWER: True. □

   (f) If $X \sim \text{Exp}(3)$, what is its moment generating function?

   ANSWER: $3/(3-t)$, for $t < 3$. □

2. Suppose that Y is a random variable with p.d.f.

$$f_Y(y) \;=\; 4y^3, \;\; 0 \le y \le 1.$$

Further suppose that the conditional p.d.f. of X given Y is

$$f(x|y) \;=\; 2x/y^2, \;\; 0 \le x \le y \le 1.$$

Show your work on the following questions.

(a) Find $f(x, y)$.

ANSWER: $f(x, y) = f(x|y)f_Y(y) = 8xy$, $0 \le x \le y \le 1$. □

(b) Find $f_X(x)$.

ANSWER: $f_X(x) = \int_x^1 f(x, y)\, dy = 4x(1 - x^2)$, $0 \le x \le 1$. □

(c) Use Part 2b to find $\mathsf{E}[X]$.

ANSWER: $\mathsf{E}[X] = \int_0^1 x f_X(x)\, dx = 8/15$. □

(d) Find $\mathsf{E}[X|y]$.

ANSWER: $\mathsf{E}[X|y] = \int_0^y x f(x|y)\, dx = 2y/3$, $0 \le y \le 1$. □

(e) Use Part 2d to find $\mathsf{E}[X]$. (This should match your answer in Part 2b.)

ANSWER: $\mathsf{E}[X] \;=\; \mathsf{E}[\mathsf{E}[X|Y]] \;=\; \int_0^1 \mathsf{E}[X|y] f_Y(y)\, dy \;=\; \int_0^1 (8/3) y^4\, dy \;=\; 8/15$. □

(f) Find the correlation between X and Y.

ANSWER: After the usual algebra, we get $\mathsf{E}[X] \;=\; 8/15$, $\mathsf{E}[X^2] \;=\; 1/3$, $\mathsf{Var}(X) = 11/225$, $\mathsf{E}[Y] = 4/5$, $\mathsf{E}[Y^2] = 2/3$, $\mathsf{Var}(Y) = 2/75$.

Further, $\mathsf{E}[XY] = \int_0^1 \int_x^1 8x^2y^2\, dy\, dx = 4/9$, so that

$$\rho \;=\; \frac{\mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; 0.4924. \quad \square$$

3. Given the following joint p.d.f.'s (or other info), determine whether or not X and Y *must* be independent. Just answer yes or no.

   (a) $g(x, y) = \lambda^2 e^{-\lambda(x+y)}$, $x > 0$, $y > 0$.

   ANSWER: Yes. □

   (b) $f(x, y) = cxy$, $0 < y < x < 1$.

   ANSWER: No. □

   (c) $f(x, y) = c(1 + x + y)^{-2}$, $x > 0$, $y > 0$.

   ANSWER: No. □

   (d) $f(x|y) = f_Y(y)$ for all y.

   ANSWER: No. □

   (e) $\mathsf{Cov}(X, Y) = 0$.

   ANSWER: No. □

4. Suppose that $\mathsf{E}(X) = 3$, $\mathsf{E}(Y) = 2$, $\mathsf{Var}(X) = 5$, $\mathsf{Var}(Y) = 4$, and $\mathsf{Cov}(X, Y) = -2$.

   (a) Find $\mathsf{E}(2X + 3Y)$.

   ANSWER: $2\mathsf{E}[X] + 3\mathsf{E}[Y] = 12$. □

   (b) Find $\mathsf{Var}(2X + 3Y)$.

   ANSWER: $4\mathsf{Var}(X) + 9\mathsf{Var}(Y) + 2\mathsf{Cov}(2X, 3Y) = 4\mathsf{Var}(X) + 9\mathsf{Var}(Y) + 12\mathsf{Cov}(X, Y) = 32$. □

   (c) Find the correlation between X and Y.

   ANSWER: We have

$$\rho = \frac{\mathsf{Cov}(X, Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} = -0.447. \quad \square$$