ISyE 6739 — Take-Home Test #2 Solutions

Summer 2001

Do *exactly* 8 (out of 9) problems. Open notes/book. Try to take less than 3 hours.

1. (Short answer sampling distribution questions — Just write your answer.)

   (a) Find $z_{0.95}$. ANSWER: -1.645. $\diamondsuit$

   (b) Find $\Phi^{-1}(0.95)$. ANSWER: 1.645. $\diamondsuit$

   (c) If $X \sim N(1,1)$, find $\mathsf{Pr}\{X < -1\}$. ANSWER: $\mathsf{Pr}\{Z < -2\} = \Phi(-2) = 0.0228$. $\diamondsuit$

   (d) Find $\chi^2_{0.95,10}$. ANSWER: 3.94. $\diamondsuit$

   (e) If $X \sim \chi^2(4)$, find $\mathsf{Pr}\{X < 11.143\}$. ANSWER: 0.975. $\diamondsuit$

   (f) Find $t_{0.025,9}$. ANSWER: 2.26. $\diamondsuit$

   (g) If $T \sim t(7)$, find $\mathsf{Pr}\{T > 2.998\}$. ANSWER: 0.01. $\diamondsuit$

   (h) Find $F_{0.10,3,5}$. ANSWER: 3.62. $\diamondsuit$

   (i) Find $F_{0.90,5,3}$. ANSWER: $1/F_{0.10,3,5} = 0.276$. $\diamondsuit$

   (j) If $F \sim F(2,2)$, find $\mathsf{Pr}\{F > 9.00\}$. ANSWER: 0.10. $\diamondsuit$

   (k) As $n \to \infty$, what does $t_{\alpha,n}$ approach? ANSWER: z_α. $\diamondsuit$

   (l) What does "d.f." mean? ANSWER: degrees of freedom. $\diamondsuit$

2. (Short answer estimation questions — Just write your answer.)

   (a) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d., then $\bar{X}$ is unbiased for $\mathsf{E}[X_i]$. $\diamondsuit$

   (b) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d., then $\frac{n-1}{n} S^2$ is unbiased for $\mathsf{Var}(X_i)$. $\diamondsuit$

   (c) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d., then $\sqrt{\bar{X}}$ is unbiased for $\sqrt{\mathsf{E}[X_i]}$. $\diamondsuit$

   (d) If $X_1, \ldots, X_n$ are i.i.d. Pois(λ), what is a method of moments estimator of λ?

   ANSWER: Since $\mathsf{E}[X_i] = \lambda$, the MOM estimator is $\hat{\lambda} = \bar{X}$. $\diamondsuit$

   (e) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(\theta, 0)$, then $\min_{1 \le i \le n}\{X_i\}$ is unbiased for θ. $\diamondsuit$

   (f) If $X_1, \ldots, X_n$ are i.i.d. $U(\theta, 0)$, find the MLE for θ.

   ANSWER: $\hat{\theta} = \min(X_i)$. $\diamondsuit$

   (g) Suppose we have 2 estimators, T_1 and T_2, for some parameter θ. Further suppose that $\mathsf{Bias}(T_1) = 10$, $\mathsf{Var}(T_1) = 10$, $\mathsf{Bias}(T_2) = 0$, and $\mathsf{Var}(T_2) = 100$. Which estimator would you use?

   ANSWER: $\mathsf{MSE}(T_1) = \mathsf{Bias}^2(T_1) + \mathsf{Var}(T_1) = 110$. Similarly, $\mathsf{MSE}(T_2) = 100$. Thus, use T_2. $\diamondsuit$

   (h) I have generously supplied you with a penny. Flip it 100 times. Based on your flips, what is the maximum likelihood estimate you obtain for the probability of tails?

   ANSWER: $\hat{p} =$ (num tails)/100. $\diamondsuit$

   (i) TRUE or FALSE? The expected length of a two-sided $100(1-\alpha)\%$ confidence interval increases as α increases. $\diamondsuit$

(j) If $S_X^2 \sim \chi^2(3)/3$, $S_Y^2 \sim \chi^2(7)/7$, and S_X^2 and S_Y^2 are independent, what is the distribution of S_Y^2/S_X^2? ANSWER: $F(7,3)$. $\diamondsuit$

3. (Short answer hypothesis testing questions — Just write your answer.)

   (a) TRUE or FALSE? The power of a test is the probability that you reject a false null hypothesis. $\diamondsuit$

   (b) TRUE or FALSE? $\alpha + \beta$ equals the size of a test plus its power. $\diamondsuit$

   (c) TRUE or FALSE? $\alpha + \beta$ equals the sum of the probabilities of Type I and II errors. $\diamondsuit$

   (d) TRUE or FALSE? For small α (say, less than 0.20) and *any* n, we have $t_{\alpha,n} < z_\alpha$. $\diamondsuit$

   (e) If $X \sim \text{Nor}(0,1)$, $Y \sim \chi^2(5)/5$, and X and Y are independent, what is the distribution of $X/\sqrt{Y}$? ANSWER: $t(5)$. $\diamondsuit$

   (f) Suppose a drug manufacturer releases a new drug that is actually *less* effective than its current brand. What kind of error has the company just committed — Type I or II? $\diamondsuit$

   (g) TRUE or FALSE? The p-value of a test is a function of α. $\diamondsuit$

   (h) BONUS: What do the bands Dinosaur Jr. and They Might Be Giants have in common? ANSWER. They both like songs by the Zombies. $\diamondsuit$

4. Suppose that the heights of men are i.i.d. Normal(70,50); those of women are i.i.d. Normal(66,40). Find the probability that a random man is shorter than a random woman.

ANSWER:

$$
\begin{aligned}
\mathsf{Pr}(M < W) &= \mathsf{Pr}(M - W < 0) \\
&= \mathsf{Pr}\left(Z < \frac{0 - (70 - 66)}{\sqrt{50 + 40}}\right) \\
&= \mathsf{Pr}(Z < -0.422) \\
&= 1 - \Phi(0.422) \\
&= 0.337 \quad \diamondsuit
\end{aligned}
$$

5. Suppose $X_1, \ldots, X_n$ are i.i.d. with p.d.f.

$$f(x) \equiv \begin{cases} \lambda e^{-\lambda(x-2)} & \text{if } x > 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the MLE for λ.

ANSWER:

$$\begin{aligned} L(\lambda) &= \prod_{i=1}^{n} f(x_i) \\ &= \prod_{i=1}^{n} \lambda e^{-\lambda(x_i-2)} \\ &= \lambda^n \exp\{-\lambda \sum_{i=1}^{n} x_i + 2\lambda n\}. \end{aligned}$$

This implies that

$$\ell\text{n}(L) = n\ell\text{n}(\lambda) - \lambda \sum_{i=1}^{n} x_i + 2\lambda n,$$

which implies that

$$\frac{\partial}{\partial\lambda}\ell\text{n}(L) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + 2n = 0.$$

Finally, after a little algebra, we get

$$\hat{\lambda} = (\bar{X} - 2)^{-1}. \quad \diamondsuit$$

(b) What would your estimate for λ be if we observed the data $X_1 = 2.5$, $X_2 = 4.0$, and $X_3 = 3.0$?

ANSWER:

$$\hat{\lambda} = (3.167 - 2)^{-1} = 0.857. \quad \diamondsuit$$

6. Suppose 4 students take an IQ test and obtain scores of 110, 130, 100, and 120. Assuming the scores are i.i.d. normal, find a 95% confidence for the true mean.

   ANSWER: The desired confidence interval is of the form

$$\mu \ \in \ \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n}.$$

   So first, we calculate the sample mean and variance.

$$\bar{X} \ = \ 115 \qquad \text{and} \qquad S^2 \ = \ \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \ = \ 166.67.$$

   Since $t_{0.025,3} = 3.182$, we have

$$\mu \ \in \ 115 \pm 3.182\sqrt{166.67/4} \ = \ 115 \pm 20.54 \ = \ (94.46,\, 135.54). \quad \diamondsuit$$

7. Suppose that we examine standardized test scores of 25 ninth-grade students. We assume that the scores are i.i.d. normal with a *known* standard deviation of 10. Further suppose that the sample mean turns out to be 100.

   (a) Find a two-sided 95% confidence interval for the unknown mean μ.

   ANSWER:

   $$\mu \in \bar{X} \pm z_{\alpha/2}\sqrt{\sigma^2/n} \;=\; 100 \pm 1.96\sqrt{100/25} \;=\; 100 \pm 3.92. \quad \diamondsuit$$

   (b) How many observations should we take (instead of 25) if we want to obtain a confidence interval with half-length (or "error") equal to 1?

   ANSWER: We want

   $$z_{\alpha/2}\sqrt{\sigma^2/n} \;\le\; \epsilon,$$

   if and only if

   $$n \;\ge\; \frac{z_{\alpha/2}^2\sigma^2}{\epsilon^2} \;=\; \left(\frac{1.96 \times 10}{1}\right)^2 \;=\; 385.$$

   (c) Test the null hypothesis that $\mu = 110$. Use a two-sided test with $\alpha = 0.05$.

   ANSWER: We'll test $H_0 : \mu = 100 = \mu_0$ vs. $H_1 : \mu \ne 100$. Use the test statistic

   $$Z_0 \;=\; \frac{\bar{X} - \mu_0}{\sqrt{\sigma^2/n}} \;=\; \frac{100 - 110}{\sqrt{100/25}} \;=\; -5.$$

   Since $Z_0 > z_{\alpha/2} = 1.96$, we REJECT H_0. $\diamondsuit$

   (d) Referring to (c), how many observations should we take if we want the probability of a Type II error to be 0.05 when μ happens to equal 100?

   ANSWER: Using $\delta = \mu - \mu_0$, we have

   $$n \;\ge\; \frac{\sigma^2}{\delta^2}(z_{\alpha/2} + z_\beta)^2 \;=\; \frac{100}{(100-110)^2}(1.96 + 1.645)^2 \;=\; 13. \quad \diamondsuit$$

8. Suppose that we want to see if a new drug lowers cholesterol levels in people. We administer the new drug to 6 people, and we obtain the following "before" and "after" results on their cholesterol levels (CL).

person	"before" CL	"after" CL
1	240	190
2	215	170
3	190	165
4	320	250
5	250	200
6	285	240

It certainly looks like the drug helps to lower the CL a bit. Let's assume that all of the above numbers are normal and that the 6 test subjects are independent of each other.

(a) Perform a one-sided hypothesis test with $\alpha = 0.05$ to check if the mean "before" CL is greater than 200. (Only use the data in the "before" column.)

ANSWER: Test $H_0 : \mu_X \leq 200 = \mu_0$ vs. $H_1 : \mu_X > 200$. We have

$$\bar{X} = 250 \qquad \text{and} \qquad S_X^2 = 2210.$$

This gives the test statistic

$$T_0 = \frac{\bar{X} - \mu_0}{\sqrt{S_X^2/n}} = 2.605.$$

Since $T_0 > t_{\alpha,n-1} = t_{0.05,5} = 2.02$, we REJECT H_0. $\diamondsuit$

(b) Find a two-sided 95% confidence interval for the unknown variance of the "before" CL's. (Only use the data in the "before" column.)

ANSWER:

$$\frac{(n-1)S_X^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2,n-1}}.$$

Since $\chi^2_{\alpha/2,n-1} = \chi^2_{0.025,5} = 12.8$ and $\chi^2_{1-\alpha/2,n-1} = \chi^2_{0.975,5} = 0.83$, we get

$$863 \leq \sigma^2 \leq 13313. \quad \diamondsuit$$

(c) Find a two-sided 95% confidence interval for the difference in the "before" and "after" means.

ANSWER: Use a PAIRED t-test. Here

$$\bar{D} \;=\; 47.5 \qquad \text{and} \qquad S_D^2 \;=\; \frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2 \;=\; 207.5.$$

Since $t_{\alpha/2,n-1} = t_{0.025,5} = 2.57$, we have

$$\mu_D \;=\; \mu_X - \mu_Y \;\in\; \bar{D} \pm t_{\alpha/2,n-1}\sqrt{S_D^2/n} \;=\; (32.4,\, 62.6). \quad \diamondsuit$$

9. Suppose that we want to see if a new inventory policy produces more profits for a local manufacturing company. We'll simulate policy A five times and policy B ten times. Let's assume that all of the simulation runs produce normal observations, and that all of the observations are independent of each other. Suppose we obtain the following results.

policy	sample mean	sample variance
A	\$1000	1600
B	\$1200	10000

(a) Does it look like you can assume that A and B have the same variance? (Don't actually perform a hypothesis test.)

ANSWER: Sample variances aren't even close, so we can't assume the variance are equal. $\diamondsuit$

(b) Find a two-sided 95% confidence interval for the unknown difference in the mean profits obtained by policies A and B.

ANSWER: Must use the approximate confidence interval,

$$\mu_A - \mu_B \ \in \ \bar{A} - \bar{B} - t_{\alpha/2,\nu}\sqrt{\frac{S_A^2}{n_A} + \frac{S_B^2}{n_B}},$$

where the approximate d.f. is

$$\nu \ = \ \frac{(S_A^2/n_A + S_B^2/n_B)^2}{\frac{(S_A^2/n_A)^2}{n_A+1} + \frac{(S_B^2/n_B)^2}{n_B+1}} - 2 \ = \ 14.$$

This implies that

$$\mu_A - \mu_B \ \in \ -200 \pm 2.14\sqrt{1320} \ = \ -200 \pm 77.8.$$