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ISyE 6739 — Test 1 Solutions — Summer 2015

This test is 100 minutes long. You are allowed one cheat sheet.

1. (50 points) Short-Answer Questions

   (a) What is any subset of the sample space called?

   Solution: Event. □

   (b) If $\mathsf{P}(A) = 0.35$ and $\mathsf{P}(B) = 0.45$, and A and B are disjoint, find $\mathsf{P}(A \cap B)$.

   Solution: 0. □

   (c) If $\mathsf{P}(A) = 0.35$ and $\mathsf{P}(B) = 0.45$, and A and B are disjoint, find $\mathsf{P}(A \cup B)$.

   Solution: $\mathsf{P}(A) + \mathsf{P}(B) = 0.8$. □

   (d) If $\mathsf{P}(A) = 0.7$ and $\mathsf{P}(B) = 0.5$, and A and B are independent, find $\mathsf{P}(A \cap B)$.

   Solution: $\mathsf{P}(A)\mathsf{P}(B) = 0.35$. □

   (e) If $\mathsf{P}(A) = 0.7$ and $\mathsf{P}(B) = 0.5$, and A and B are independent, find $\mathsf{P}(A \cup B)$.

   Solution: $\mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A)\mathsf{P}(B) = 0.85$. □

   (f) TRUE or FALSE? $\bar{A} \cup \bar{B} = \overline{A \cap B}$.

   Solution: TRUE — this is DeMorgan's Law. □

   (g) TRUE or FALSE? If $n \geq r \geq 0$, then $\binom{n}{r} = P_{n,r}/r!$.

   Solution: TRUE. □

(h) Calculate $\binom{5}{2}$.

Solution: $\frac{5!}{2!\,3!} = 10$. $\square$

(i) Calculate $P_{9,4}$.

Solution: $9!/5! = 3024$. $\square$

(j) TRUE or FALSE? $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ for any $n \geq 0$.

Solution: TRUE. $\square$

(k) TRUE or FALSE? If A and B are disjoint, then they are independent.

Solution: FALSE. $\square$

(l) TRUE or FALSE? The events A and $\bar{A}$ always form a partition.

Solution: TRUE. $\square$

(m) TRUE or FALSE? If A and B are independent, then $\mathsf{P}(A|B) = \mathsf{P}(A \cap B)/\mathsf{P}(A)$.

Solution: FALSE. $\square$

(n) How many ways can you arrange the letters in "MISSPELLING"?

Solution: $11!/(2!\,2!\,2!) = 4989600$. $\square$

(o) If I told you to "Pay attention to your posterior", which important theorem would I be talking about?

Solution: Bayes Theorem (which deals with posterior probabilities). $\square$

(p) TRUE or FALSE? $\mathsf{P}(A \cup B \cup C) \leq \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C)$.

Solution: TRUE — Just look at a Venn diagram. $\square$

(q) Who is the best stats teacher ever in the entire universe?

Solution: So easy, I won't even bother writing the answer. □

2. (10 points) A box contains 3 red sox and 2 yellows. Two sox are selected randomly without replacement.

(a) What is this experiment's sample space?

Solution: $\{RR, RY, YR, YY\}$. □

(b) Suppose X denotes the number of yellow sox selected. What are the possible values of X?

Solution: $X = 0, 1, 2$. □

(c) Calculate the probability that $X = 1$.

Solution: $\mathsf{P}(RY) + \mathsf{P}(YR) = \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = 0.6$. □

3. (10 points) Everybody in Buffoonalo, NY participates in at least one of the following sports: bowling, skiing, and aerobics. In particular, 60% of the people bowl, 60% ski, and 50% do aerobics; 40% bowl and ski; 20% bowl and do aerobics; and 30% ski and do aerobics.

(a) What proportion of the people bowl *or* ski?

Solution: $\mathsf{P}(B \cup S) = \mathsf{P}(B) + \mathsf{P}(S) - \mathsf{P}(B \cap S) = 0.6 + 0.6 - 0.4 = 0.8$. □

(b) What proportion of the people participate in all three sports?

Solution: Since everyone in Buffoonalo does something, we have

$$\begin{aligned} 1 &= \mathsf{P}(B \cup S \cup A) \\ &= \mathsf{P}(B) + \mathsf{P}(S) + \mathsf{P}(A) - \mathsf{P}(B \cap S) - \mathsf{P}(B \cap A) - \mathsf{P}(S \cap A) + \mathsf{P}(B \cap S \cap A) \\ &= 0.6 + 0.6 + 0.5 - 0.4 - 0.2 - 0.3 + \mathsf{P}(B \cap S \cap A), \end{aligned}$$

so that $\mathsf{P}(B \cap S \cap A) = 0.2$. $\square$

(c) What proportion only do aerobics (and nothing else)?

Solution: You can find $\mathsf{P}(\bar{B} \cap \bar{S} \cap A) = 0.2$ via a careful Venn diagram (as we've done in class). $\square$

4. (10 points) An electronic assembly consists of two subsystems, say A and B. Suppose we have the following information:

   - P(B fails) = 0.6
   - P(A and B fail) = 0.3
   - P(A fails but B doesn't fail) = 0.1

   Find the probability that B fails given that A fails.

   Solution: By the law of total probability,

   $$\mathsf{P}(\text{A fails}) \;=\; \mathsf{P}(\text{A and B fail}) + \mathsf{P}(\text{A fails and B doesn't fail}) \;=\; 0.4.$$

   Therefore,

   $$\mathsf{P}(\text{B fails} \mid \text{A fails}) \;=\; \frac{\mathsf{P}(\text{A and B fail})}{\mathsf{P}(\text{A fails})} \;=\; 3/4. \quad \square$$

5. (10 points) Pick 7 cards from a standard deck. Find the probability of getting exactly three pairs and one junk card.

   Solution:
   Number of ways to pick 3 pairs = $\binom{13}{3}$.
   Number of ways to pick suits for 3 pairs = $\binom{4}{2}^3$.
   Number of ways to junk card = 40.

   Thus, the desired probability is

   $$\frac{\binom{13}{3}\binom{4}{2}^3 40}{\binom{52}{7}} \;\approx\; 0.0185. \quad \square$$

6. (10 points) There are 6 married couples are in a room. Suppose that we randomly pair up each of the men with one of the women (for conversation purposes only).

   (a) How many possible pairings are there?

   Solution: 36. $\square$

   Actually, if you interpreted the question to mean how many *sets* of pairings are there, I guess I could accept $6! = 720$. $\square$

   (b) What is the probability that *every* husband and wife will be matched correctly?

   Solution: $1/6! = 1/720$. $\square$

   (c) What is the probability that at least one of the pairs will be an actual husband–wife pair?

   Solution: This is essentially an envelope problem with $n = 6$. Therefore, the desired probability is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} = \frac{91}{144} = 0.6319 \approx 1 - \frac{1}{e}. \quad \square$$

7. (50 points) Still More Short-Answer Questions...

   (a) What does the "m" in "p.m.f." mean?

   Solution: mass. $\square$

   (b) Suppose X has p.d.f. $f(x) = x^3$, $0 \le x \le c$. Find c.

   Solution: $\int_0^c f(x)\,dx = 1$ implies $c = 4^{1/4} = \sqrt{2} = 1.414$. $\square$

   (c) Suppose that X is discrete with $f(-2) = 0.2$, $f(0) = 0.5$, and $f(4) = 0.3$. Find $F(1.2)$, where $F(x)$ is the c.d.f. of X.

   Solution: $F(1.2) = \mathsf{P}(X \le 1.2) = f(-2) + f(0) = 0.7$. $\square$

(d) Suppose X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. Find $\mathsf{P}(3/4 \le X \le 2)$.

Solution: $\mathsf{P}(3/4 \le X \le 2) = \mathsf{P}(3/4 \le X \le 1) = \int_{3/4}^{1} 2x\,dx = 7/16.$ $\square$

(e) Suppose X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. Find $\mathsf{P}(0 \le X \le 1/2 \,|\, 1/4 \le X \le 3/4)$.

Solution:

$$\begin{aligned}
\mathsf{P}(0 \le X \le 1/2 \,|\, 1/4 \le X \le 3/4) &= \frac{\mathsf{P}(0 \le X \le 1/2 \;\cap\; 1/4 \le X \le 3/4)}{\mathsf{P}(1/4 \le X \le 3/4)} \\
&= \frac{\mathsf{P}(1/4 \le X \le 1/2)}{\mathsf{P}(1/4 \le X \le 3/4)} \\
&= \frac{\int_{1/4}^{1/2} f(x)\,dx}{\int_{1/4}^{3/4} f(x)\,dx} \\
&= 3/8. \quad \square
\end{aligned}$$

(f) If $f(y) = 2y$, $0 < y < 1$, find $\mathsf{E}[Y]$.

Solution: $\int_0^1 y f(y)\,dy = 2/3.$ $\square$

(g) If $f(y) = 2y$, $0 < y < 1$, find $\mathsf{E}[1/Y^2]$.

Solution: $\int_0^1 (1/y^2) f(y)\,dy = \infty$. (Oops! Didn't mean to trick you!) $\square$

(h) If $\mathsf{E}[X] = 3$ and $\mathsf{E}[X^2] = 10$, find $\mathsf{Var}(X)$.

Solution: $\mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 1.$ $\square$

(i) TRUE or FALSE? $\mathsf{Var}(X) \ge 0$ for any random variable X.

Solution: Let $\mu = \mathsf{E}[X]$. Then, by definition, $\mathsf{Var}(X) = \mathsf{E}[(X - \mu)^2]$. This is the expected value of something that cannot be negative. Therefore, the answer is TRUE. $\square$

(j) If $\mathsf{E}[X] = 3$ and $\mathsf{Var}(X) = 7$, find $\mathsf{E}[-2X - 5]$.

Solution: $\mathsf{E}[-2X - 5] = -2\mathsf{E}[X] - 5 = -11$. $\Box$

(k) If $\mathsf{E}[X] = 3$ and $\mathsf{Var}(X) = 7$, find $\mathsf{Var}(-2X - 5)$.

Solution: $\mathsf{Var}(2X - 3) = 4\mathsf{Var}(X) = 28$. $\Box$

(l) If $\mathsf{P}(X = 0) = 0.4$ and $\mathsf{P}(X = 1) = 0.6$, name the distribution of X (including any parameter(s)).

Solution: Bernoulli(0.6). $\Box$

(m) If $\mathsf{P}(X = 0) = 0.4$ and $\mathsf{P}(X = 1) = 0.6$, find $\mathsf{E}[\ell n(X + 1)]$.

Solution: By LOTUS, $\sum_x \ell n(x + 1) f(x) = \ell n(1) f(0) + \ell n(2) f(1) = \ell n(2) \cdot 0.6 = 0.4159$. $\Box$

(n) What result says that $\mathsf{P}(|X - \mathsf{E}[X]| \geq 1) \leq \mathsf{Var}(X)$? for any random variable X?

Solution: Chebychev's inequality (with $\epsilon = 1$). $\Box$

(o) Suppose that the probability that GT wins any football game in 0.7, and that all games are (somehow) independent. What is the probability that GT will win exactly 3 out of its next 4 games?

Solution: Let X be the number of games won $\sim$ Binomial$(4, 0.7)$. Then $\mathsf{P}(X = 3) = \binom{4}{3}(0.7)^3(0.3)^1 = 0.4116$. $\Box$

(p) Consider a lightbulb whose lifetime is exponential with a mean of 1 year. What's the probability that the bulb will live at least two years before failing?

Solution: Let X be the lifetime $\sim$ Exp(1). Then $\mathsf{P}(X > 2) = \int_2^\infty e^{-x}\, dx = e^{-2} = 0.1353$. $\Box$