

## ISyE 6739 — Test #1 Solutions

### Summer 2009

This test is open notes, open books. You have *exactly 75 minutes*.

#### 1. Basic Probability Questions (3 points each)

- (a) What is any subset of a sample space called?

**Solution:** An event.  $\square$

- (b) A box contains 3 red marbles and 2 blues. Two marbles are selected randomly without replacement. What is the probability that both are blue?

**Solution:**  $\frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$ .  $\square$

- (c) TRUE or FALSE?  $\Pr(A \cap B) = \Pr(A|B) \Pr(A)$ .

**Solution:** False.  $\Pr(A \cap B) = \Pr(A|B) \Pr(B)$ .  $\square$

- (d) The probability that it will rain tomorrow is 0.6. The probability that it will be windy tomorrow is 0.5. Suppose that rain and wind are independent. What's the probability that it will be *both* rainy and windy tomorrow?

**Solution:** By independence,  $\Pr(R \cap W) = \Pr(R) \Pr(W) = 0.3$ .  $\square$

- (e) In a certain population, 50% of the people jog, 60% play tennis, and 70% swim. Further, 30% jog and play tennis, 40% jog and swim, 50% play tennis and swim, and 20% do all three. What's the probability that a random person from this population only jogs (and does nothing else)?

**Solution:** Draw a Venn diagram, and work from the inside out. You'll eventually find that  $\Pr(J \cap \bar{T} \cap \bar{S}) = 0$ .  $\square$

- (f) How many ways can you scramble the letters in "STATISTICS"?

**Solution:**  $\frac{10!}{3!3!2!1!1!}$ .  $\square$

- (g) A 4-sided die is thrown 5 times. Find  $\Pr(\text{Each face appears at least once})$ .

**Solution:** Similar to what we did on an old HW, consider the generic form of a toss meeting the requirement of all faces appearing at least once — AABCD.

The number of ways to choose A is 4.

The number of ways to place the pair of A's is  $\binom{5}{2}$ .

The number of ways to scramble BCD is 3!

And the total number of ways to toss 5 dice is  $4^5$ .

Therefore the desired probability is  $4 \binom{5}{2} 3! / 4^5$ .  $\square$

- (h) Pick 6 cards from a standard deck. Find the probability of four aces.

**Solution:**  $\binom{4}{4} \binom{48}{2} / \binom{52}{6}$ .  $\square$

- (i) Suppose you have 3 people in the room. Assuming all 365 days of the year are equally likely, find the probability that at least two of these people have the same birthday.

**Solution:**  $\Pr(\text{at least 2}) = 1 - \Pr(\text{all different}) = 1 - \frac{365 \cdot 364 \cdot 363}{365 \cdot 365 \cdot 365}$ .  $\square$

- (j) Kobe Bryant is great free throw shooter. In fact, his shots are independent and all have probability 0.8 of being successful. Suppose he takes 10 shots in the next game. What's the probability that he makes exactly 8 of them?

**Solution:** Let  $X$  denote the number of successes. Thus,  $X \sim \text{Bin}(10, 0.8)$ . This implies that  $\Pr(X = 8) = \binom{10}{8} (0.8)^6 (0.2)^2$ .  $\square$

- (k) TRUE or FALSE? If  $A_1, A_2, \dots, A_n$  form a partition of the sample space, then for any event  $B$ , we have  $\Pr(B) = \sum_{i=1}^n \Pr(A_i \cap B)$ .

**Solution:** True.  $\square$

- (l) TRUE or FALSE?  $P_{n,k} = \binom{n}{k} k!$  for all  $(n, k)$ .

**Solution:** True.  $\square$

## 2. Questions on Random Variables (3 points each)

- (a) Consider the p.d.f.
- $f(x) = 3x^2$
- ,
- $0 \leq x \leq c$
- . Find the appropriate value of
- $c$
- .

**Solution:** Setting  $\int_R f(x) dx = \int_0^c 3x^2 dx = 1$ , we immediately get  $c = 1$ .  $\square$ 

- (b) If
- $X$
- has p.d.f.
- $f(x) = 3e^{-3x}$
- for
- $x \geq 0$
- , find the c.d.f.
- $F(x)$
- .

**Solution:** This is the  $\text{Exp}(3)$  p.d.f., so that  $F(x) = 1 - e^{-3x}$ ,  $x \geq 0$ .  $\square$ 

- (c) If
- $X$
- is a Bernoulli(0.7) random variable, find
- $\Pr(X = 0)$
- .

**Solution:**  $\Pr(X = 0) = 1 - p = 0.3$ .  $\square$ 

- (d) If
- $X$
- has p.d.f.
- $f(x) = x/2$
- ,
- $0 \leq x \leq 2$
- , find the conditional probability
- $\Pr(0 \leq X \leq 1/2 | 0 \leq X \leq 1)$
- .

**Solution:**

$$\begin{aligned}
\Pr(0 \leq X \leq 1/2 | 0 \leq X \leq 1) &= \frac{\Pr(0 \leq X \leq 1/2 \cap 0 \leq X \leq 1)}{\Pr(0 \leq X \leq 1)} \\
&= \frac{\Pr(0 \leq X \leq 1/2)}{\Pr(0 \leq X \leq 1)} \\
&= \frac{\int_0^{1/2} \frac{x}{2} dx}{\int_0^1 \frac{x}{2} dx} \\
&= 1/4. \quad \square
\end{aligned}$$

- (e) If
- $X$
- has p.d.f.
- $f(x) = x/2$
- ,
- $0 \leq x \leq 2$
- , find
- $\mathbf{E}[X]$
- .

**Solution:**  $\mathbf{E}[X] = \int_R x f(x) dx = \int_0^2 (x^2/2) dx = 4/3$ .  $\square$ 

- (f) If
- $X$
- has p.d.f.
- $f(x) = x/2$
- ,
- $0 \leq x \leq 2$
- , find
- $\mathbf{Var}(X)$
- .

**Solution:** First of all,  $\mathbf{E}[X^2] = \int_R x^2 f(x) dx = \int_0^2 (x^3/2) dx = 2$ . Then the previous problem implies  $\mathbf{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 2 - \frac{16}{9} = \frac{2}{9}$ .  $\square$

(g) If  $X$  has p.d.f.  $f(x) = x/2$ ,  $0 \leq x \leq 2$ , find  $E[1/X]$ .

**Solution:** By the Law of the Unconscious Statistician,  $E[1/X] = \int_{\mathcal{R}} (1/x)f(x) dx = \int_0^2 (1/2) dx = 1$ .  $\square$

(h) If  $X \sim \text{Exp}(1)$ , find  $E[4X + 2]$ .

**Solution:**  $4E[X] + 2 = 6$ .  $\square$

(i) If  $E[X] = 7$  and  $\text{Var}(X) = 10$ , find  $\text{Var}(4X + 2)$ .

**Solution:**  $16\text{Var}(X) = 160$ .  $\square$

(j) The *moment generating function* of a random variable  $X$  is defined as  $M_X(t) \equiv E[e^{tX}]$ . It can be shown that  $\frac{d}{dt}M_X(t)|_{t=0} = E[X]$  (you don't have to prove this). Anyway, suppose  $X$  has m.g.f.  $M_X(t) = pe^t + q$ . Find the expected value of  $X$ .

**Solution:**  $\frac{d}{dt}M_X(t)|_{t=0} = \frac{d}{dt}(pe^t + q)|_{t=0} = pe^t|_{t=0} = p$ . (This is great, since it turns out that  $X \sim \text{Bern}(p)$ .)  $\square$

(k) If  $X \sim \text{Pois}(2)$ , find  $\Pr(X \geq 2)$ .

**Solution:**  $\Pr(X \geq 2) = 1 - \Pr(X = 0 \text{ or } 1) = 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - 3e^{-2} = 0.594$ .  $\square$

(l) The second central moment of a random variable  $X$  is more-commonly known as the \_\_\_\_\_ [fill in the blank].

**Solution:** variance.  $\square$

3. (9 points) If a person has cancer, a certain test for cancer is positive (i.e., detects the cancer) 90% of the time. If the person does *not* have cancer, the test is positive 5% of the time. In the general population, 1/100 of the people have the type of cancer that this test detects. If a person is selected at random and the test is positive, find the probability that does not have cancer. **[Show your work here.]**

**Solution:** Let  $C$  = cancer and  $P$  = positive. Then Bayes implies

$$\Pr(\bar{C}|P) = \frac{\Pr(P|\bar{C}) \Pr(\bar{C})}{\Pr(P|\bar{C}) \Pr(\bar{C}) + \Pr(P|C) \Pr(C)} = \frac{(0.05)(0.99)}{(0.05)(0.99) + (0.9)(0.01)} = \frac{11}{13}. \quad \square$$

4. (9 points) Suppose  $X \sim \text{Unif}(0, 1)$ , i.e.,  $f(x) = 1$ ,  $0 < x < 1$ . Find the p.d.f. of  $Y = e^{X+2}$ .

**Solution:** The c.d.f. of  $Y$  is

$$\begin{aligned} G(y) &= \Pr(Y \leq y) = \Pr(e^{X+2} \leq y) = \Pr(X \leq \ln(y) - 2) \\ &= \int_0^{\ln(y)-2} f(x) dx = \ln(y) - 2. \end{aligned}$$

Thus, the p.d.f. is  $g(y) = \frac{d}{dy}G(y) = \frac{1}{y}$ ,  $e^2 \leq y \leq e^3$ . (Make sure you know where the limits come from.)  $\square$

5. (9 points) Pick 8 cards from a standard deck. Find the probability of getting exactly one 3-of-a-kind, 2 pairs, and one junk card, e.g.,  $A\clubsuit, A\diamond, A\heartsuit, 3\clubsuit, 3\diamond, 10\diamond, 10\spadesuit, 5\spadesuit$ .

**Solution:** Similar to what we've done a few times in class, consider the generic form of a hand meeting the stated requirement — AAABBCCD.

Pick the rank of the 3-of-a-kind A — 13 ways

Pick two ranks for the two pairs (B and C) —  $\binom{12}{2}$  ways.

Pick the suits of the 3-of-a-kind A —  $\binom{4}{3}$  ways.

Pick the suits of the pair B —  $\binom{4}{2}$  ways; pick the suits of the pair C —  $\binom{4}{2}$  ways.

Pick the junk card D — 40 ways.

Thus, the desired probability is  $\frac{13 \binom{12}{2} \binom{4}{3} \binom{4}{2}^2 40}{\binom{52}{8}}$ .  $\square$