

ISyE 6739 — Test #1 Solutions

Summer 2005

This test is open notes, open books. You have *exactly 75 minutes*.

1. (10 points) A box contains 5 red marbles and 3 blues. Two marbles are selected randomly. What is the probability that one is red and one is blue if we sample...

(a) With replacement?

Solution: $\frac{5}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{5}{8} = \frac{15}{32}$. \square

(b) Without replacement?

Solution: $\frac{\binom{5}{1} \binom{3}{1}}{\binom{8}{2}} = \frac{15}{28}$. \square

2. (10 points) In a certain population, 70% of the people like The Beatles, 60% like The Rolling Stones, and 50% like The Zombies. In addition, 50% like The Beatles and The Stones, 40% like The Beatles and The Zombies, 40% like The Stones and The Zombies, and 35% like all three music groups. What percentage of people like at least one of the three groups?

Solution:

$$\Pr(B) + \Pr(S) + \Pr(Z) - \Pr(BS) - \Pr(BZ) - \Pr(SZ) + \Pr(BSZ) = 0.85. \quad \square$$

3. (10 points) How many ways can you scramble the letters in "TENNESSEE"?

Solution: $\frac{9!}{1!4!2!2!} = 3780$. \square

4. (10 points) A 5-sided die is thrown 6 times. Find the probability that each face appears at least once.

Solution: Find all the ways to scramble tosses of the form AABCDE. Then the desired probability is

$$\frac{5 \binom{6}{2} 4!}{5^6} = 0.1152. \quad \square$$

5. (10 points) Pick 6 cards from a standard deck. Find the probability of getting exactly two pairs.

Solution: Find all the ways to select the cards. Then the desired probability is

$$\frac{[\text{choose two pairs}][\text{choose two remaining cards}]}{\binom{52}{6}} = \frac{\binom{13}{2} \binom{4}{2}^2 \binom{11}{2} 4^2}{\binom{52}{6}}. \quad \square$$

6. (10 points) It rains in beautiful Syracuse 60% of the time. If it is raining, then Tommy will wear his hat with probability 0.6. If it is not raining, then Tommy will wear his hat with probability 0.2. If Tommy is seen wearing his hat, what's the probability that it's raining?

Solution: By Bayes,

$$\Pr(R|H) = \frac{\Pr(H|R)\Pr(R)}{\Pr(H|R)\Pr(R) + \Pr(H|\bar{R})\Pr(\bar{R})} = \frac{(0.6)(0.6)}{(0.6)(0.6) + (0.2)(0.4)} = \frac{9}{11}. \quad \square$$

7. (20 points) Consider the continuous random variable X having p.d.f.

$$f(x) = \begin{cases} 4x^3 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} .$$

(a) Find the c.d.f. $F(x)$.

Solution:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^4 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} . \quad \square$$

(b) Find $\Pr(0 \leq X \leq 1/3)$.

Solution: $F(1/3) = 1/81$. \square

(c) Find $\Pr(0 \leq X \leq 1/3 | 0 \leq X \leq 2/3)$.

Solution: $\frac{\Pr(0 \leq X \leq 1/3)}{\Pr(0 \leq X \leq 2/3)} = \frac{(1/3)^4}{(2/3)^4} = \frac{1}{16}$. \square

(d) Find $E[X]$.

Solution: $\int_0^1 4x^4 dx = 4/5$. \square

(e) Find $\text{Var}(X)$.

Solution: $E[X^2] = \int_0^1 4x^5 dx = 2/3$. So $\text{Var}(X) = 2/75$. \square

(f) Find $E[4X + 2]$.

Solution: $4E[X] + 2 = 5.2$. \square

(g) Find $\text{Var}(4X + 2)$.

Solution: $16\text{Var}(X) = 32/75$. \square

(h) Find $E[(X - 1)^2]$.

Solution: $E[X^2] - 2E[X] + 1 = 1/15$. \square

8. (20 points) Short answer questions — Just write your answer.

(a) TRUE or FALSE? If A and B are disjoint, then A and B are independent.

Solution: False. \square

(b) TRUE or FALSE? $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Solution: True. \square

(c) TRUE or FALSE? The Law of the Unconscious Statistician states that $E[g(X)] = g(E[X])$ for any reasonable function X .

Solution: False. \square

(d) TRUE or FALSE? If $M_X(t)$ is the m.g.f. of X , then $\text{Var}(X) = \left[\frac{d^2}{dt^2} M_X(t) - \left(\frac{d}{dt} M_X(t) \right)^2 \right]_{t=0}$.

Solution: True. \square

(e) $0! = ?$

Solution: 1. \square

(f) If $X \sim \text{Bern}(0.7)$, find $\Pr(X = 1)$.

Solution: 0.7. \square

(g) If $X \sim \text{Bin}(5, 0.7)$, find $\Pr(X = 4)$.

Solution: $\binom{5}{4} (0.7)^4 (0.3)^1 = 0.360$. \square

(h) If $X \sim \text{Pois}(4)$, find $\Pr(X = 5)$.

Solution: $\frac{e^{-4} 4^5}{5!} = 0.156$. \square