ISyE 6739 — Test #1 Solutions

Summer 2002

(Updated 6/27/05)

This test is open notes, open books. You have *exactly 90 minutes.*

1. A box contains 20 light bulbs, 5 of which are defective. Suppose we select 10 bulbs from the box without replacement.

   (a) Find the probability that exactly 3 are defective.

   Solution:

$$\frac{\binom{5}{3}\binom{15}{7}}{\binom{20}{10}}. \quad \square$$

   (b) Find the probability that at most 2 are defective.

   Solution:

$$\sum_{i=0}^{2} \frac{\binom{5}{i}\binom{15}{10-i}}{\binom{20}{10}}. \quad \square$$

2. Suppose that A, B, and C are three *independent* events such that $\mathsf{Pr}(A) = 1/4$, $\mathsf{Pr}(B) = 1/3$, and $\mathsf{Pr}(C) = 1/2$.

   (a) Find the probability that *none* of the events will occur.

   Solution: By independence, we have

$$\mathsf{Pr}(\bar{A} \cap \bar{B} \cap \bar{C}) \;=\; \mathsf{Pr}(\bar{A})\mathsf{Pr}(\bar{B})\mathsf{Pr}(\bar{C}) \;=\; (3/4)(2/3)(1/2) \;=\; 1/4. \quad \square$$

   (b) Find the probability that exactly one of the events will occur.

Solution: Now we want

$$\begin{aligned}
&\mathsf{Pr}(A\cap\bar{B}\cap\bar{C})+\mathsf{Pr}(\bar{A}\cap B\cap\bar{C})+\mathsf{Pr}(\bar{A}\cap\bar{B}\cap C)\\
&= \mathsf{Pr}(A)\mathsf{Pr}(\bar{B})\mathsf{Pr}(\bar{C})+\mathsf{Pr}(\bar{A})\mathsf{Pr}(B)\mathsf{Pr}(\bar{C})+\mathsf{Pr}(\bar{A})\mathsf{Pr}(\bar{B})\mathsf{Pr}(C)\\
&= 11/24. \quad \square
\end{aligned}$$

3. If a person has cancer, a certain test for cancer is positive (i.e., detects the cancer) 95% of the time. If the person does *not* have cancer, the test is positive 5% of the time. In the general population, 1/500 of the people have the type of cancer that this test detects. If a person is selected at random and the test is positive, find the probability that he indeed has cancer.

   Solution: Let P denote a positive test and C = cancer. Then Bayes implies

$$\begin{aligned}
\mathsf{Pr}(C|P) &= \frac{\mathsf{Pr}(P|C)\mathsf{Pr}(C)}{\mathsf{Pr}(P|C)\mathsf{Pr}(C)+\mathsf{Pr}(P|\bar{C})\mathsf{Pr}(\bar{C})}\\
&= \frac{0.95(1/500)}{0.95(1/500)+0.05(499/500)}\\
&= 0.037. \quad \square
\end{aligned}$$

4. Pick 8 cards from a standard deck. Find the probability of getting exactly two 3-of-a-kinds and one pair, e.g., $A\clubsuit$, $A\diamondsuit$, $A\heartsuit$, $3\clubsuit$, $3\diamondsuit$, $3\spadesuit$, $10\diamondsuit$, $10\spadesuit$.

   Solution: Using standard counting arguments, the desired probability is

$$\frac{\text{(choose the pair)(choose the triples)}}{\text{(choose 8 cards)}} = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{2}\binom{4}{3}^2}{\binom{52}{8}}. \quad \square$$

5. Quickie Probability Questions — Just Write Your Answer.

   (a) $\mathsf{Pr}(A\cap B)=\mathsf{Pr}(A)\mathsf{Pr}(B)$ implies that A and B are…

   Solution: Independent. $\square$

   (b) $P_{6,3}=$

   Solution: $6!/3! = 120$. $\square$

(c) TRUE or FALSE? $\mathsf{Pr}(A|B) = \mathsf{Pr}(A)$ implies that A and B are independent.

Solution: True. $\square$

(d) TRUE or FALSE? If $n \geq k$, then $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.

Solution: Using some standard algebra, we find that

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n!}{k!}\left[\frac{1}{(n-k)!} + \frac{k}{(n-k+1)!}\right] \\ &= \frac{n!}{k!(n-k+1)!}[n-k+1+k] \\ &= \binom{n+1}{k}.\end{aligned}$$

So the answer is True. $\square$

(e) If X is a continuous random variable with p.d.f. $f(x) = 4x^3$ for $0 < x < c$, find c.

Solution: 1. $\square$

(f) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{E}[X]$.

Solution: $\int_0^1 2x^2\,dx = 2/3$. $\square$

(g) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{Var}(X)$.

Solution: $\mathsf{E}[X^2] = 1/2$, so $\mathsf{Var}(X) = 1/18$. $\square$

(h) TRUE or FALSE? If $A \subset B$, then $\mathsf{Pr}(A) \leq \mathsf{Pr}(B)$.

Solution: True. $\square$

6. Consider the continuous random variable Y having p.d.f.

$$f(x) = \begin{cases} cx & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find c.

Solution: 2. □

(b) Find $\mathsf{Pr}(0 \le X \le 0.5)$.

Solution: $\int_0^{1/2} 2x\,dx = 1/4$. □

(c) Find $\mathsf{Pr}(0 \le X \le 0.25 |\, 0 \le X \le 0.5)$.

Solution: $\mathsf{Pr}(0 \le X \le 0.25)/\mathsf{Pr}(0 \le X \le 0.5) = 1/4$. □

(d) Find $\mathsf{E}[X]$.

Solution: $2/3$. □

(e) Find $\mathsf{Var}(X)$.

Solution: $1/18$. □

(f) Find $\mathsf{E}[2X - 2]$.

Solution: $-2/3$. □

(g) Find $\mathsf{Var}(2X - 2)$.

Solution: $2/9$. □

(h) Use Chebychev's inequality to get an upper bound on $\mathsf{Pr}(|X - \mathsf{E}[X]| > 0.25)$.

Solution: $\mathsf{Var}(X)/\epsilon^2 = (1/18)/(1/4)^2 = 8/9$. □

7. Suppose $X \sim \text{Unif}(0,1)$, i.e., $f(x) = 1$, $0 < x < 1$.

   (a) Find the p.d.f. of $Y = e^{3X+2}$.

   Solution: In class, we found that $g(y) = 1/(3y)$, $e^2 < y < e^5$. □

   (b) Find the p.d.f. of $Y = -\ell\text{n}(X) + 1$.

   Solution: In class, we found that $g(y) = e^{1-y}$, $y > 1$. □