ISyE 6739 — Take-Home Test #1 Solutions
Summer 2001

Do *exactly* 10 (out of 12) problems. Open notes/book. Try to take less than 3 hours.

1. A box contains 2 red sox, 4 blue sox, and 3 yellows. Two sox are selected randomly without replacement.

   (a) What is this experiment's sample space?

   Solution: $S = \{RR, RB, RY, BR, BB, BY, YR, YB, YY\}$. $\diamondsuit$

   (b) Suppose X denotes the number of yellow sox selected. What are the possible values of X?

   Solution: X can equal 0,1,2. $\diamondsuit$

   (c) Calculate the probability that $X = 0$.

   Solution:

   $$\Pr(X = 0) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}. \quad \diamondsuit$$

2. Everybody in Syracuse, NY participates in at least one of the following sports: bowling, skiing, and aerobics. In particular, 60% of the people bowl, 65% ski, and 65% do aerobics; 35% bowl and ski; 35% bowl and do aerobics; and 50% ski and do aerobics. What proportion of the people participate in all three sports?

   Solution: $\Pr(B) = 0.6$, $\Pr(S) = 0.65$, $\Pr(A) = 0.65$, $\Pr(B \cap S) = 0.35$, $\Pr(B \cap A) = 0.35$, $\Pr(S \cap A) = 0.5$, and $\Pr(B \cup S \cup A) = 1$ (since everyone participates).

   So by the principle of inclusion-exclusion, we have

   $$\begin{aligned} 1 &= \Pr(B \cup S \cup A) \\ &= \Pr(B) + \Pr(S) + \Pr(A) - \Pr(B \cap S) - \Pr(B \cap A) - \Pr(S \cap A) + \Pr(B \cap S \cap A), \end{aligned}$$

   which implies that $\Pr(B \cap S \cap A) = 0.3$. $\diamondsuit$

3. An electronic assembly consists of two subsystems, say A and B. Suppose we have the following information:

   - Pr(B fails) = 0.5
   - Pr(A and B fail) = 0.3
   - Pr(A fails but B doesn't fail) = 0.3

Find the probability that B fails given that A fails.

Solution: $\Pr(B) = 0.5$, $\Pr(A \cap B) = 0.3$, $\Pr(A \cap \bar{B}) = 0.3$.

Thus,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A \cap B)}{\Pr(A \cap B) + \Pr(A \cap \bar{B})} = 1/2. \quad \diamondsuit$$

4. Pick 6 cards from a standard deck. Find the probability of getting exactly three pairs.

   Solution:

$$\Pr(3 \text{ pairs}) = \frac{\binom{13}{3}\binom{4}{2}^3}{\binom{52}{6}}. \quad \diamondsuit$$

5. (Short answer questions — Just write your answer.)

   (a) The set of all outcomes of an experiment is called ____________.

   Solution: The sample space. $\diamondsuit$

   (b) Any subset of the above set is called ____________.

   Solution: An event. $\diamondsuit$

   (c) If A and B are disjoint, then $\Pr(A \cup B) = ?$

   Solution: $\Pr(A) + \Pr(B)$. $\diamondsuit$

   (d) If $\Pr(A) = 0.7$ and $\Pr(B) = 0.6$, and A and B are independent, then

   i. $\Pr(A \cap B) = ?$

   Solution: $\Pr(A)\Pr(B) = 0.42$. $\diamondsuit$

   ii. $\Pr(A \cup B) = ?$

   Solution: $\Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.88$. $\diamondsuit$

   (e) TRUE or FALSE? $\bar{A} \cup \bar{B} = \overline{A \cap B}$

   Solution: TRUE $\diamondsuit$

   (f) TRUE or FALSE? $C_{n,r} = P_{n,r}/r!$

   Solution: TRUE $\diamondsuit$

   (g) $\binom{5}{0} = ?$

   Solution: 1. $\diamondsuit$

   (h) $\binom{100}{97} = ?$

   Solution: $\frac{(100)(99)(98)}{(3)(2)(1)} = 161700$. $\diamondsuit$

(i) $P_{15,3} = ?$
Solution: $\frac{15!}{12!} = 2730.$ $\diamondsuit$

6. Box A has 4 blue and 6 red sox. Box B has 5 blues and 5 reds. Let's roll a die. If the die's outcome is odd, then a sock from A is selected; if the outcome is even, a sock from B is selected. Suppose that a blue sock is selected. What is the probability that the die toss was even?

Solution: By Bayes' rule,

$$\begin{aligned}\Pr(\text{Even} \mid \text{Blue}) &= \frac{\Pr(\text{Even})\Pr(\text{Blue} \mid \text{Even})}{\Pr(\text{Even})\Pr(\text{Blue} \mid \text{Even}) + \Pr(\text{Odd})\Pr(\text{Blue} \mid \text{Odd})} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{5}} = 5/9. \ \diamondsuit\end{aligned}$$

7. Consider the continuous random variable Y having p.d.f.

$$f(y) = \begin{cases} c|y|^3 & \text{if } -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) What does "p.d.f." mean?
Solution: probability density function. $\diamondsuit$

(b) Find c.
Solution: $c = 2$ $\diamondsuit$

(c) Find $\mathsf{Pr}(-1 \le Y \le 0)$.
Solution: $1/2$ $\diamondsuit$

(d) Find $\mathsf{Pr}(0 \le Y \le 0.5 | \, 0 \le Y \le 1)$.
Solution: $1/16$ $\diamondsuit$

(e) Find $\mathsf{Pr}(0 \le Y \le 0.5 | -1 \le Y \le -0.5)$.
Solution: 0 $\diamondsuit$

(f) Find $\mathsf{E}[Y]$.
Solution: 0 $\diamondsuit$

(g) Find $\mathsf{Var}(Y)$.
Solution: $2/3$ $\diamondsuit$

(h) Find $\mathsf{E}[3Y - 2]$.
Solution: -2 $\diamondsuit$

(i) Find $\mathsf{Var}(3Y - 2)$.
Solution: 6 $\diamondsuit$

(j) Use Chebychev's inequality to get an upper bound on $\mathsf{Pr}(|Y - \mathsf{E}[Y]| > 0.5)$.
Solution: $8/3$ $\diamondsuit$

8. TRUE-FALSE Questions. X and Y must be independent if

(a) $f(x|y) = f_Y(y)$ for all y.
Solution: FALSE $\diamondsuit$

(b) $\mathsf{Cov}(X, Y) = 0$.
Solution: FALSE $\diamondsuit$

(c) $f(x, y) = cy$, $0 < x < y < 1$.
Solution: FALSE $\diamondsuit$

(d) $f(x, y) = cy^2/(1 + x^3)$, $0 < x < 1$, $1 < y < 3$.
Solution: TRUE $\diamondsuit$

(e) $\mathsf{E}(XY) = \mathsf{E}(X) \cdot \mathsf{E}(Y)$.
Solution: FALSE $\diamondsuit$

9. Suppose $f(x, y) = cx$, $0 < y < x < 1$.

(a) Find c.
Solution: 3 $\diamondsuit$

(b) Find $\mathsf{Pr}(X < 0.5 \text{ and } Y > 0.5)$.
Solution: 0 $\diamondsuit$

(c) Find the p.d.f. of Y.
Solution: $f_Y(y) = \frac{3}{2}(1 - y^2)$, $0 < y < 1$. $\diamondsuit$

(d) Find the conditional p.d.f. of X given that $Y = y$.
Solution: $f(x|y) = \frac{2x}{1-y^2}$, $0 < y < x < 1$. $\diamondsuit$

(e) Find $\mathsf{E}(X|Y = y)$.
Solution: $\frac{2}{3} \cdot \frac{1+y+y^2}{1+y}$. $\diamondsuit$

10. Suppose that $\mathsf{E}(X) = 3$, $\mathsf{E}(Y) = 2$, $\mathsf{Var}(X) = 5$, $\mathsf{Var}(Y) = 4$, and $\mathsf{Cov}(X, Y) = -2$.

(a) Find $\mathsf{E}(2X + 3Y)$.
Solution: $\mathsf{E}(2X + 3Y) = 2\mathsf{E}(X) + 3\mathsf{E}(Y) = 12$. $\diamondsuit$

(b) Find $\mathsf{Var}(2X + 3Y)$.
Solution: $\mathsf{Var}(2X + 3Y) = 4\mathsf{Var}(X) + 9\mathsf{Var}(Y) + 2(2)(3)\mathsf{Cov}(X, Y) = 32$. $\diamondsuit$

11. If the m.g.f. of X is $M_X(t) = e^{2t^2}$, find $\mathsf{E}(X)$.

    Solution:

$$\mathsf{E}(X) \;=\; \frac{d}{dt}\,M_X(t)\bigg|_{t=0} \;=\; \frac{d}{dt}\,e^{2t^2}\bigg|_{t=0} \;=\; \frac{d}{dt}\,4te^{2t^2}\bigg|_{t=0} \;=\; 0. \;\diamondsuit$$

12. Suppose that a light bulb has a lifetime that is exponentially distributed with a mean of 1000 hours. Suppose the bulb has already survived 3000 hours. What's the probability that it will survive another 1000 hours?

    Solution: By the memoryless property,

$$\Pr(X \ge 4000 \,|\, X \ge 3000) \;=\; \Pr(X \ge 1000) \;=\; e^{-\lambda x} \;=\; e^{-1} \;=\; 0.368. \;\diamondsuit$$