ISyE 6739 — Practice Test #2 Solutions
Summer 2009

This test is open notes, open books. You have *exactly 90 minutes*. Just do the best you can, and good luck!

1. Short Answer Questions.

   (a) Suppose X has p.d.f. $f(x) = 5x^4$, $0 \le x \le 1$. Find $\mathsf{E}[2X-5]$.

   ANSWER: $2\mathsf{E}[X] - 5 = -10/3$.

   (b) If X again has p.d.f. $f(x) = 5x^4$, $0 \le x \le 1$, find $\mathsf{Var}(2X-5)$.

   ANSWER: $4\mathsf{Var}(X) = 0.0793$.

   (c) Suppose X can equal 1 or 2, each with probability 1/2. Find $\mathsf{E}[\ell\mathrm{n}(X)]$.

   ANSWER: $\frac{1}{2}\,\ell\mathrm{n}(2) = 0.347$.

   (d) TRUE or FALSE? $\mathsf{E}[X^2] \ge (\mathsf{E}[X])^2$.

   ANSWER: True.

   (e) TRUE or FALSE? If $\mathsf{Cov}(X,Y) = 0$, then X and Y are independent.

   ANSWER: False.

   (f) Suppose that X and Y are independent Exponential(λ) random variables. Find $\mathsf{Var}(XY)$ (yup — the variance of the product).

   ANSWER: $\mathsf{E}[X^2Y^2] - (\mathsf{E}[XY])^2 = \mathsf{E}[X^2]\mathsf{E}[Y^2] - (\mathsf{E}[X])^2(\mathsf{E}[Y])^2 = 3/\lambda^4$.

   (g) What does "i.i.d." mean?

   ANSWER: independent and identically distributed.

(h) Bonus Question: If you were Dutch, what would "Q-65" mean to you?

ANSWER: They later became Golden Earring.

2. Suppose X and Y are two random variables such that $\mathsf{E}[X] = 3$, $\mathsf{Var}(X) = 4$, $\mathsf{E}[Y] = -2$, $\mathsf{Var}(Y) = 9$, and $\mathsf{Cov}(X, Y) = -2$.

(a) Find the correlation between $2X$ and $3Y$.

ANSWER:

$$\mathsf{Corr}(2X, 3Y) \;=\; \frac{\mathsf{Cov}(2X, 3Y)}{\sqrt{\mathsf{Var}(2X)\mathsf{Var}(3Y)}} \;=\; \frac{\mathsf{Cov}(X, Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; -1/3.$$

(b) Find $\mathsf{Var}(X - Y)$.

ANSWER: $\mathsf{Var}(X) + \mathsf{Var}(Y) - 2\mathsf{Cov}(X, Y) = 17$.

3. (10 points) Suppose X has p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$. What is the p.d.f. of $Y = \sqrt{X}$?

ANSWER: The c.d.f. of Y is

$$\begin{aligned} G(y) &= \Pr(Y \le y) \;=\; \Pr(\sqrt{X} \le y) \\ &= \Pr(X \le y^2) \;=\; \int_0^{y^2} 3x^2\,dx \\ &= y^6. \end{aligned}$$

Thus, the required p.d.f. is $g(y) = 6y^5$, $0 \le y \le 1$.

4. Suppose that $f(x,y) = cxy$ if $0 < y < x < 1$.

   (a) Find c.

   ANSWER: Use $1 = \int_0^1 \int_0^x cxy\,dy\,dx$ to get $c = 8$.

   (b) Find $f_X(x)$.

   ANSWER: $\int_0^x 8xy\,dy = 4x^3$, $0 < x < 1$.

   (c) Find $\mathsf{Corr}(X,Y)$.

   ANSWER: Using the usual methods, we have

$$\mathsf{E}[X] \;=\; \frac{4}{5}, \quad \mathsf{Var}(X) \;=\; \frac{2}{75}, \quad \mathsf{E}[Y] \;=\; \frac{8}{15}, \quad \mathsf{Var}(Y) \;=\; \frac{11}{225},$$

   and

$$\mathsf{E}[XY] \;=\; \int_0^1 \int_0^x 8x^2y^2\,dy\,dx \;=\; \frac{4}{9}.$$

   Thus,

$$\rho \;=\; \frac{\mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; 0.492.$$

   (d) Find $f(y|x)$.

   ANSWER: $f(x,y)/f_X(x) = 2y/x^2$, $0 < y < x < 1$.

   (e) Use the above answer to find $\mathsf{E}[Y]$.

   ANSWER: First,

$$\mathsf{E}[Y|x] \;=\; \int_{\Re} y f(y|x)\,dy \;=\; \int_0^x \frac{2y^2}{x^2}\,dy \;=\; \frac{2x}{3}.$$

   Then

$$\mathsf{E}[Y] \;=\; \mathsf{E}[\mathsf{E}[Y|X]] \;=\; \int_{\Re} \mathsf{E}[Y|x] f_X(x)\,dx \;=\; \frac{8}{15}.$$

5. Suppose X is a discrete random variable whose only possible values are the non-negative integers. Define the *probability generating function* of X by

$$g_X(s) \equiv \mathsf{E}[s^X] = \sum_{k=0}^{\infty} s^k \Pr(X = k).$$

   (a) Prove $\mathsf{E}[X] = \frac{d}{ds}\, g_X(s)|_{s=1}$.

   ANSWER:

$$\frac{d}{ds}\, g_X(s)|_{s=1} = \frac{d}{ds}\, \mathsf{E}[s^X]\Big|_{s=1} = \mathsf{E}\left[\frac{d}{ds} s^X\right]\Bigg|_{s=1} = \mathsf{E}\Big[X s^{X-1}\Big]\Big|_{s=1} = \mathsf{E}[X].$$

   (b) If $X \sim \text{Pois}(\lambda)$, find $g_X(s)$. (You may need the fact that $\sum_{k=0}^{\infty} y^k/k! = e^y$.)

   ANSWER:

$$g_X(s) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{\lambda(s-1)}.$$

   (c) Suppose that $X \sim \text{Pois}(\lambda)$. Use the above two parts of the problem to find $\mathsf{E}[X]$.

   ANSWER:

$$\mathsf{E}[X] = \frac{d}{ds}\, g_X(s)|_{s=1} = \lambda e^{\lambda(s-1)}\Big|_{s=1} = \lambda.$$

6. (Short answer sampling distribution questions — Just write your answer.)

   (a) Find $z_{0.025}$.

   ANSWER: 1.96.

   (b) Find $\Phi(1.85)$.

   ANSWER: 0.9678.

   (c) If $X \sim N(2, 4)$, find $\mathsf{Pr}\{X < 4\}$.

   ANSWER: $\Pr\left(\frac{X-2}{2} < \frac{4-2}{2}\right) = \Phi(1) = 0.8413$.

(d) Find $\chi^2_{0.025,15}$.

ANSWER: 27.49.

(e) If $X \sim \chi^2(3)$, find $\mathsf{Pr}\{X > 7.815\}$.

ANSWER: 0.05.

(f) Find $t_{0.05,15}$.

ANSWER: 1.753.

(g) If $T \sim t(7)$, find $\mathsf{Pr}\{T < 2.365\}$.

ANSWER: 0.975.

(h) Find $F_{0.025,3,4}$.

ANSWER: 9.98.

(i) Find $F_{0.975,4,3}$.

ANSWER: $1/9.98 = 0.1002$.

(j) If $F \sim F(5,3)$, find $\mathsf{Pr}\{F < 14.88\}$.

ANSWER: 0.975.

(k) BONUS: Where are the Cucumbers from?

ANSWER: Hoboken, NJ.

7. (Short answer estimation questions — Just write your answer.)

(a) Find the sample standard deviation of 7, 12, and 1.

ANSWER: $\sqrt{30.33} = 5.51$.

(b) If $X_1, \ldots, X_n$ are i.i.d. Bern(p), what is the MLE of p?

ANSWER: $\bar{X}$.

(c) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, where μ and σ^2 are unknown, then S^2 is unbiased for σ^2.

ANSWER: True.

(d) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, where μ and σ^2 are unknown, then S^2 is the MLE for σ^2.

ANSWER: False.

(e) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. Exp(λ), then $1/\bar{X}$ is a MOM estimator for λ.

ANSWER: True.

(f) If $X_1, \ldots, X_n$ are i.i.d. with $\mathsf{E}[X_i] = 3$ and $\mathsf{Var}(X_i) = 10$, find $\mathsf{E}[S^2]$.

ANSWER: 10.

(g) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(0, \theta)$, then $\max_{1 \le i \le n}\{X_i\}$ is unbiased for θ.

ANSWER: False.

(h) TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(0, \theta)$, then $\max_{1 \le i \le n}\{X_i\}$ is the MLE for θ.

ANSWER: True.

(i) What does "MSE" mean?

ANSWER: mean-squared error.

(j) Suppose we have 2 estimators, T_1 and T_2, for some parameter θ. Further suppose that $\mathsf{Bias}(T_1) = 5$, $\mathsf{Var}(T_1) = 10$, $\mathsf{Bias}(T_2) = 0$, and $\mathsf{Var}(T_2) = 50$. Which estimator would you use?

ANSWER: $\mathsf{MSE}(T_1) = 35$, $\mathsf{MSE}(T_2) = 50$, so choose T_1.

(k) BONUS: Who is Dr. Paul Arnold?

ANSWER: He's one of the original Zombies.

8. Suppose X_1, X_2, X_3 are i.i.d. normal with mean 1 and variance 1. What is $\mathsf{Pr}\{2X_1 - X_2 - X_3 > 0\}$?

   ANSWER: $2X_1 - X_2 - X_3 \sim \text{Nor}(2-1-1, 4+1+1) \sim \text{Nor}(0,6)$, so an easy symmetry argument says that the desired probability is 1/2.

9. Suppose $X_1, \ldots, X_n$ are i.i.d. with p.d.f.

$$f(x) \equiv \begin{cases} (\gamma+1)x^\gamma & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Find the MLE for γ.

   ANSWER:

$$L(\gamma) = \prod_{i=1}^n f(x_i) = (\gamma+1)^n (\prod_{i=1}^n x_i)^\gamma.$$

   Thus,

$$\ell\text{n}(L) = n\ell\text{n}(\gamma+1) + \gamma\,\ell\text{n}(\prod_{i=1}^n x_i),$$

   and so

$$\frac{d}{d\gamma}\ell\text{n}(L) = \frac{n}{\gamma+1} + \ell\text{n}(\prod_{i=1}^n x_i) = 0.$$

   Solving for γ gives

$$\hat{\gamma} = \frac{-n}{\ell\text{n}(\prod_{i=1}^n x_i)} - 1.$$

   (b) What would your estimate for γ be if we observed the data $X_1 = 0.3$, $X_2 = 0.8$, and $X_3 = 0.5$?

   ANSWER: Plugging the three observations into the above equation, we get $\hat{\gamma} = 0.415$.