ISyE 6739 — Practice Test #1 Solutions

Summer 2009

This test is open notes, open books.

1. A box contains 5 red marbles and 3 blues. Two marbles are selected randomly. What is the probability that one is red and one is blue if we sample...

   (a) With replacement?

   Solution: $\frac{5}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{5}{8} = \frac{15}{32}$. □

   (b) Without replacement?

   Solution: $\frac{\binom{5}{1}\binom{3}{1}}{\binom{8}{2}} = \frac{15}{28}$. □

2. In a certain population, 70% of the people like The Beatles, 60% like The Rolling Stones, and 50% like The Zombies. In addition, 50% like The Beatles and The Stones, 40% like The Beatles and The Zombies, 40% like The Stones and The Zombies, and 35% like all three music groups. What percentage of people like at least one of the three groups?

   Solution:

$$\mathsf{Pr}(B) + \mathsf{Pr}(S) + \mathsf{Pr}(Z) - \mathsf{Pr}(BS) - \mathsf{Pr}(BZ) - \mathsf{Pr}(SZ) + \mathsf{Pr}(BSZ) \;=\; 0.85. \;\square$$

3. How many ways can you scramble the letters in "TENNESSEE"?

   Solution: $\frac{9!}{1!4!2!2!} = 3780$. □

4. (10 points) A 5-sided die is thrown 6 times. Find the probability that each face appears at least once.

Solution: Find all the ways to scramble tosses of the form AABCDE. Then the desired probability is

$$\frac{5\binom{6}{2}4!}{5^6} = 0.1152. \quad \square$$

5. Pick 6 cards from a standard deck. Find the probability of getting exactly two pairs.

Solution: Find all the ways to select the cards. Then the desired probability is

$$\frac{[\text{choose two pairs}][\text{choose two remaining cards}]}{\binom{52}{6}} = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{2}4^2}{\binom{52}{6}}. \quad \square$$

6. It rains in beautiful Syracuse 60% of the time. If it is raining, then Tommy will wear his hat with probability 0.6. If it is not raining, then Tommy will wear his hat with probability 0.2. If Tommy is seen wearing his hat, what's the probability that it's raining?

Solution: By Bayes,

$$\mathsf{Pr}(R|H) = \frac{\mathsf{Pr}(H|R)\mathsf{Pr}(R)}{\mathsf{Pr}(H|R)\mathsf{Pr}(R) + \mathsf{Pr}(H|\bar{R})\mathsf{Pr}(\bar{R})} = \frac{(0.6)(0.6)}{(0.6)(0.6) + (0.2)(0.4)} = \frac{9}{11}. \quad \square$$

7. Consider the continuous random variable X having p.d.f.

$$f(x) = \begin{cases} 4x^3 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the c.d.f. $F(x)$.

Solution:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^4 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}. \quad \Box$$

(b) Find $\mathsf{Pr}(0 \le X \le 1/3)$.

Solution: $F(1/3) = 1/81$. $\Box$

(c) Find $\mathsf{Pr}(0 \le X \le 1/3 | \, 0 \le X \le 2/3)$.

Solution: $\dfrac{\mathsf{Pr}(0 \le X \le 1/3)}{\mathsf{Pr}(0 \le X \le 2/3)} = \dfrac{(1/3)^4}{(2/3)^4} = \dfrac{1}{16}$. $\Box$

(d) Find $\mathsf{E}[X]$.

Solution: $\int_0^1 4x^4 \, dx = 4/5$. $\Box$

(e) Find $\mathsf{Var}(X)$.

Solution: $\mathsf{E}[X^2] = \int_0^1 4x^5 \, dx = 2/3$. So $\mathsf{Var}(X) = 2/75$. $\Box$

(f) Find $\mathsf{E}[4X + 2]$.

Solution: $4\mathsf{E}[X] + 2 = 5.2$. $\Box$

(g) Find $\mathsf{Var}(4X + 2)$.

Solution: $16\mathsf{Var}(X) = 32/75$. $\Box$

(h) Find $\mathsf{E}[(X - 1)^2]$.

Solution: $\mathsf{E}[X^2] - 2\mathsf{E}[X] + 1 = 1/15$. $\Box$

8. Short answer questions — Just write your answer.

   (a) TRUE or FALSE? If A and B are disjoint, then A and B are independent.

   Solution: False. $\square$

   (b) TRUE or FALSE? $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

   Solution: True. $\square$

   (c) TRUE or FALSE? The Law of the Unconscious Statistician states that $\mathsf{E}[g(X)] = g(\mathsf{E}[X])$ for any reasonable function X.

   Solution: False. $\square$

   Solution: True. $\square$

   (d) $0! = ?$

   Solution: 1. $\square$

   (e) If $X \sim \text{Bern}(0.7)$, find $\mathsf{Pr}(X = 1)$.

   Solution: 0.7. $\square$

   (f) If $X \sim \text{Bin}(5, 0.7)$, find $\mathsf{Pr}(X = 4)$.

   Solution: $\binom{5}{4}(0.7)^4(0.3)^1 = 0.360.$ $\square$

   (g) If $X \sim \text{Pois}(4)$, find $\mathsf{Pr}(X = 5)$.

   Solution: $\dfrac{e^{-4}4^5}{5!} = 0.156.$ $\square$

   (h) The *moment generating function* of a random variable X is defined as $M_X(t) \equiv \mathsf{E}[e^{tX}]$. It can be shown that $\frac{d}{dt}M_X(t)|_{t=0} = \mathsf{E}[X]$ (you don't have to prove this). Anyway, suppose that a certain random variable X's m.g.f. is

$M_X(t) = \frac{2}{2-t}$ (for $t < 2$). Find the expected value of X.

Solution:

$$\mathsf{E}[X] = \frac{d}{dt}\frac{2}{2-t}\Big|_{t=0} = \frac{2}{(2-t)^2}\Big|_{t=0} = 1/2.$$

(It turns out that this was the m.g.f. of the Exp(2) distribution.) □

9. Short-Answer Questions

(a) What is the set of all possible outcomes of an experiment called?

Solution: Sample space. □

(b) Suppose Pr(I jog today) = 0.4, Pr(I watch TV) = 0.8, and Pr(I do both) = 0.3. What's the probability that I'll do either or both activities?

Solution: Let J = I jog today, T = I watch TV. Then

$$\begin{aligned}\mathsf{Pr}(\text{I'll do either or both}) &= \mathsf{Pr}(J \cup T) \\ &= \mathsf{Pr}(J) + \mathsf{Pr}(T) - \mathsf{Pr}(J \cap T) \\ &= 0.4 + 0.8 - 0.3 \\ &= 0.9. \quad \square\end{aligned}$$

(c) If $\mathsf{Pr}(A) = 0.3$ and $\mathsf{Pr}(B) = 0.5$, and A and B are disjoint, find $\mathsf{Pr}(A \cap \bar{B})$.

Solution: Since A and B are disjoint, $A \subseteq \bar{B}$ (as is easily verified by a Venn diagram). Thus $A \cap \bar{B} = A$. Thus, $\mathsf{Pr}(A \cap \bar{B}) = \mathsf{Pr}(A) = 0.3$. □

(d) If $\mathsf{Pr}(A) = 0.3$ and $\mathsf{Pr}(B) = 0.5$, and A and B are disjoint, find $\mathsf{Pr}(A \cup \bar{B})$.

Solution: As in the previous question, $A \subseteq \bar{B}$. Thus, $A \cup \bar{B} = \bar{B}$, and so $\mathsf{Pr}(A \cup \bar{B}) = \mathsf{Pr}(\bar{B}) = 1 - P(B) = 1 - 0.5 = 0.5$. □

(e) Suppose that $\mathsf{Pr}(A) = \mathsf{Pr}(B) = \mathsf{Pr}(C) = 0.5$ and that A, B, and C are all independent. Find $\mathsf{Pr}(A \cap B \cap C)$.

Solution: Since A, B, and C are all independent, the probability of their intersection is just the product of their probabilities. That is, $\mathsf{Pr}(A \cap B \cap C) =$

$\mathsf{Pr}(A)\mathsf{Pr}(B)\mathsf{Pr}(C) = 0.5^3 = 0.125.$ □

(f) Suppose that $\mathsf{Pr}(A) = \mathsf{Pr}(B) = \mathsf{Pr}(C) = 0.5$ and that A, B, and C are all independent. Find $\mathsf{Pr}(A \cup B \cup C)$.

Solution: We know that in general for three events,

$$\mathsf{Pr}(A \cup B \cup C) = \mathsf{Pr}(A) + \mathsf{Pr}(B) + \mathsf{Pr}(C) - \mathsf{Pr}(A \cap B) - \mathsf{Pr}(A \cap C) - \mathsf{Pr}(B \cap C) + \mathsf{Pr}(A \cap B \cap C).$$

Now since the three events are all independent, the probability of the intersections of two or three of them is just the product of the respective probabilities. Thus, we have

$$\mathsf{Pr}(A \cup B \cup C) = 0.5 + 0.5 + 0.5 - 0.5^2 + 0.5^2 + 0.5^2 + 0.5^3 = 7/8. \quad \square$$

(g) TRUE or FALSE? $\mathsf{Pr}(A) = 0$ implies that A is the null set.

Solution: FALSE. For example, if you pick a random number between 0 and 1, the probability of a particular outcome is zero. □

(h) TRUE or FALSE? If $\mathsf{Pr}(A \cap B \cap C) = \mathsf{Pr}(A)\mathsf{Pr}(B)\mathsf{Pr}(C)$, then A, B, and C are independent.

Solution: FALSE. It could be the case that $\mathsf{Pr}(A \cap B) \neq \mathsf{Pr}(A)\mathsf{Pr}(B)$. □

(i) Calculate $\binom{5}{3}$.

Solution: 10. □

(j) TRUE or FALSE? $\binom{500}{200} = \binom{500}{300}$.

Solution: TRUE. In general, $\binom{n}{k} = \binom{n}{n-k}$. □

(k) Consider the set of letters $\{a, b, c, d, e, f, g, h, i, j\}$. How many distinct 4-letter "words" can you make from this set of 10 letters?

Solution: There are two possible answers, depending on your interpretation. With repetitions allowed: 10^4. □

With repetitions not allowed: $10 \times 9 \times 8 \times 7 = P_{10,4} = 5040$. □

(l) TRUE or FALSE? If A and B are independent, then $\mathsf{Pr}(A|B) = \mathsf{Pr}(B)$.

Solution: FALSE. In fact, if A and B were independent, then we would have $P(A|B) = P(A)$. □

(m) How many ways can you arrange the letters in "ARKANSAS"?

Solution: $\frac{8!}{3!2!} = 3360$. □

(n) TRUE or FALSE? $\mathsf{Pr}(A \cap B \cap C) = \mathsf{Pr}(A|B \cap C)\mathsf{Pr}(B|C)\mathsf{Pr}(C)$.

Solution: TRUE. This follows because

$$\begin{aligned}\mathsf{Pr}(A|B \cap C)\mathsf{Pr}(B|C)\mathsf{Pr}(C) &= \frac{\mathsf{Pr}(A \cap B \cap C)}{\mathsf{Pr}(B \cap C)} \times \frac{\mathsf{Pr}(B \cap C)}{\mathsf{Pr}(C)} \times \mathsf{Pr}(C) \\ &= \mathsf{Pr}(A \cap B \cap C). \quad \square\end{aligned}$$

(o) Find Pr(the sum of three fair dice is at most 4).

Solution: The only favorable events are 111, 112, 121 and 211. The total number of possible outcomes is $6^3 = 216$. Therefore, the desired probability is $\frac{4}{216}$. □

10. Short-Answer Questions on RV's.

(a) What does "RV" stand for?

Solution: random variable. □

(b) What does the "d" in "p.d.f." stand for?

Solution: density. □

(c) Suppose that X is continuous with p.d.f. $f(x) = cx^2$ for $0 < x < 1$. Find c.

Solution: Since $f(x)$ is a p.d.f., we have

$$1 \;=\; \int_0^1 f(x)\,dx \;=\; \frac{cx^3}{3}\bigg|_0^1 \;=\; \frac{c}{3},$$

which implies $c = 3$. $\square$

(d) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{Pr}(0 \le X \le 0.5)$.

Solution: We have

$$P(0 \le x \le 0.5) = \int_0^{0.5} 2x\,dx = x^2\Big|_0^{0.5} = 0.25. \quad \square$$

(e) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{E}[X]$.

Solution: We have

$$\mathsf{E}[X] = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\bigg|_0^1 = \frac{2}{3}. \quad \square$$

(f) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{E}[X^3]$.

Solution: By the Law of the Unconscious Statistician, we have

$$\mathsf{E}[X^3] = \int_0^1 x^3 \cdot 2x\,dx = \int_0^1 2x^3\,dx = \frac{2x^4}{5}\bigg|_0^1 = \frac{2}{5}. \quad \square$$

(g) Suppose X has p.m.f. $f(x) = e^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \ldots$. Name the distribution of X.

Solution: Poisson(λ). $\square$

(h) Suppose that X has an Exponential distribution with parameter $\lambda = 3$. Find $\mathsf{E}[2X + 1]$.

Solution: $\mathsf{E}[2X + 1] = 2\mathsf{E}[X] + 1 = 2 \times \frac{1}{3} + 1 = \frac{5}{3}$. $\square$

(i) TRUE or FALSE? If X is continuous, then its c.d.f. is the integral of its p.d.f.

Solution: TRUE. $\square$

(j) TRUE or FALSE? If X is continuous and always positive, then $\mathsf{E}[X] = \int_0^\infty \mathsf{Pr}(X > x)\,dx$.

Solution: TRUE. This follows because

$$\begin{aligned}\int_0^\infty \mathsf{Pr}(X > x)\,dx &= \int_0^\infty \int_x^\infty f(t)\,dt\,dx \\ &= \int_0^\infty \int_0^t f(t)\,dx\,dt \\ &= \int_0^\infty t\,f(t)\,dt \\ &= \mathsf{E}[X]. \quad \square\end{aligned}$$

11. Toss 10 fair dice and let X denote the number of times a "3" appears.

(a) Name the distribution of X.

Solution: Binomial$(10, \frac{1}{6})$. $\square$

(b) Is X continuous or discrete?

Solution: Discrete. $\square$

(c) What's the probability that "3"' comes up exactly twice?

Solution:

$$\mathsf{Pr}(X = 2) = \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8. \quad \square$$

12. Still More Short-Answer Questions

(a) Assuming that all 365 birthdays are created equal, what's the probability that at least two people will have a matching birthday if there are 4 people in the room?

Solution: $\mathsf{Pr}(\text{at least two}) = 1 - \mathsf{Pr}(\text{none})$. Thus, we have

$$\mathsf{Pr}(\text{at least two people will have a match}) \;=\; 1 - \frac{365 \times 364 \times 363 \times 362}{365^4}. \quad \Box$$

(b) Suppose we have written 4 letters, but we have randomly inserted them in the 4 corresponding envelopes. (Oops!) What is the probability that at least one of the letters will be in its proper envelope?

Solution: $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$. (This is the classic envelope problem, as described in class.) $\Box$

(c) Choose 6 cards from a standard deck. What's the probability that you will get 3 pairs?

Solution: The total number of hands is $\binom{52}{6}$.
There are $\binom{13}{3}$ ways to pick the three ranks of the pairs.
For each pair, there are $\binom{4}{2}$ ways of picking the suits for the two cards that constitute the pair.

Thus the desired probability is

$$\frac{\binom{13}{3}\binom{4}{2}^3}{\binom{52}{6}}. \quad \Box$$

13. The Short-Answer Questions Keep on Coming!

(a) Suppose X has p.d.f. $f(x) = 3x^2$, $0 < x < 1$. Find $\mathsf{Pr}(0 \le X \le 1/4 \,|\, 0 \le X \le 1/2)$.

Solution: $\mathsf{Pr}(0 \le X \le 1/4 \,|\, 0 \le X \le 1/2)$
$=$
$$\frac{\mathsf{Pr}(0 \le X \le 1/4 \cap 0 \le X \le 1/2)}{\mathsf{Pr}(0 \le X \le 1/2)} = \frac{\mathsf{Pr}(0 \le X \le 1/4)}{\mathsf{Pr}(0 \le X \le 1/2)} = \frac{\int_0^{1/4} 3x^2 dx}{\int_0^{1/2} 3x^2 dx} = \frac{x^3|_0^{1/4}}{x^3|_0^{1/2}} = 1/8 \quad \Box$$

(b) If $f(x) = x/2$, $0 < x < 2$, find $\mathsf{E}[X]$.

Solution:

$$\mathsf{E}[X] = \int_R x\, f(x)\, dx = \int_0^2 x \cdot \frac{x}{2}\, dx = \left.\frac{x^3}{6}\right|_0^2 = \frac{4}{3} \quad \Box$$

(c) If $f(x) = x/2$, $0 < x < 2$, find $\mathsf{E}[1/X]$.

Solution:

$$\mathsf{E}\left[\frac{1}{X}\right] = \int_R \frac{1}{x} f(x)\, dx = \int_0^2 \frac{1}{x} \cdot \frac{x}{2}\, dx = 1 \quad \square$$

(d) If $\mathsf{E}[X] = 2$ and $\mathsf{E}[X^2] = 10$, find $\mathsf{Var}(2X - 1)$.

Solution: $\mathsf{Var}(2X - 1) = 4\mathsf{Var}(X) = 4(\mathsf{E}[X^2] - \mathsf{E}[X]^2) = 4(10 - 4) = 24 \quad \square$

(e) True or False? If $\mathsf{E}[X] = \mu$, then $\mathsf{E}[(X - \mu)^2] = \mathsf{E}[X^2] - \mu^2$ for any random variable X.

Solution: TRUE (from class notes) $\square$

(f) The lifetime of a certain type of transistor can be described by an exponential random variable X having p.d.f. $f(x) = 0.01e^{-0.01x}$ for $x \geq 0$. Find the c.d.f. of $Y = X^2$.

Solution: $G(y) = \mathsf{Pr}(Y \leq y) = \mathsf{Pr}(X^2 \leq y) = \mathsf{Pr}(X \leq \sqrt{y}) = 1 - e^{-0.01\sqrt{y}} \quad \square$