

ISyE 6739 — Summer 2017
Homework #6 (Modules 3.1–3.3) — Solutions

1. (Hines et al., 5–2. binomial.) Six independent trips to the moon are planned, each of which has estimated success probability 0.95. What's the probability that at least 5 will be successful?

Solution:

$$\begin{aligned}\Pr(X \geq 5) &= \sum_{x=5}^6 \binom{6}{x} (0.95)^x (0.05)^{6-x} \\ &= 6(0.95)^5(0.05) + (0.95)^6 \\ &= 0.9672. \quad \diamond\end{aligned}$$

2. (Hines et al., 5–6. binomial m.g.f.) Find the mean and variance of the binomial using the m.g.f.

Solution:

$$\begin{aligned}M_X(t) &= \mathbb{E}[e^{tX}] = \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= (pe^t + q)^n, \quad \text{where } q = 1 - p.\end{aligned}$$

$$\mathbb{E}[X] = M'_X(0) = [n(pe^t + q)^{n-1}pe^t]_{t=0} = np.$$

$$\begin{aligned}\mathbb{E}[X^2] &= M''_X(0) = np[e^t(n-1)(pe^t + q)^{n-2}(pe^t) + (pe^t + q)^{n-1}e^t]_{t=0} \\ &= (np)^2 - np^2 + np.\end{aligned}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq. \quad \diamond$$

3. (Hines et al., 5–9. geometric.) The probability of a successful firing of a cruise missile is 0.95. Assuming independent tests, what's the prob that the first failure occurs with the fifth missile?

Solution: $\Pr(X = 5) = (0.95)^4(0.05) = 0.0407. \quad \diamond$

4. (Hines et al., 5–30. Poisson.) Phone calls arrive at a switchboard according to a Pois(10/hour) process. The current system can handle up to 20 calls in an hour without becoming overloaded. What's the probability of an overload in the next hour?

Solution:

$$\begin{aligned} \Pr(X > 20) = \Pr(X \geq 21) &= \sum_{x=21}^{\infty} \frac{e^{-10}(10)^x}{x!} \\ &= 1 - \Pr(X \leq 20) = 1 - \sum_{x=0}^{20} \frac{e^{-10}(10)^x}{x!} \\ &= 0.002. \quad \diamond \end{aligned}$$

5. (Hines et al., 6–13. exponential.) The time to failure of a TV is exponential with a mean of 3 years. A company offers insurance for the first year of usage. On what percentage of policies will the company have to pay claims?

Solution: Let $X =$ Life Length.

$$E(X) = \frac{1}{\lambda} = 3 \quad \Rightarrow \quad \lambda = \frac{1}{3},$$

so

$$\Pr(X < 1) = 1 - e^{-1/3} = 0.283.$$

Thus, 28.3% of policies result in a claim. \diamond

6. (Hines et al., 6–16. exponential.) A transistor has an exponential time-to-failure distribution with a mean-time-to-failure of 20,000 hours. Suppose that the transistor has already lasted 20,000 hours. What's the probability that it fails by 30,000 hours?

Solution:

$$\Pr(X > x+s | X > x) = \Pr(X > s) = \Pr(X > 10000) = e^{-10000/20000} = 0.6064,$$

so $\Pr(X < 30000 | X > 20000) = 0.3936. \quad \diamond$

7. (Hines et al., 7–1(a)–(e). normal.) Suppose Z is standard normal. Find

- (a) $\Pr(0 < Z < 2)$.
- (b) $\Pr(-1 < Z < 1)$.
- (c) $\Pr(Z < 1.65)$.
- (d) $\Pr(Z > -1.96)$.
- (e) $\Pr(|Z| > 1.5)$.

Solution:

- (a) $\Pr(0 \leq Z \leq 2) = \Phi(2) - \Phi(0) = 0.97725 - 0.5 = 0.47725$.
- (b) $\Pr(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68268$.
- (c) $\Pr(Z \leq 1.65) = \Phi(1.65) = 0.95053$.
- (d) $\Pr(Z \geq -1.96) = \Phi(1.96) = 0.9750$.
- (e) $\Pr(|Z| \geq 1.5) = 2[1 - \Phi(1.5)] = 0.1336$. \diamond

8. (Hines et al., 7-3(a). normal.) Find c such that $\Phi(c) = 0.94062$.

Solution: From the back of the book, $c = \Phi^{-1}(0.94062) = 1.56$. \diamond

9. (Hines et al., 7-5(a). normal.) If $X \sim N(80, 10^2)$, find $\Pr(X < 100)$.

Solution: $\Pr(X \leq 100) = \Phi\left(\frac{100 - 80}{10}\right) = \Phi(2) = 0.97725$. \diamond

10. (Hines et al., 7-7. normal.) A manager requires job applicants to take a test and score a 500. The test scores are normally distributed with a mean of 485 and standard deviation of 30. What percent of applicants pass?

Solution: $\Pr(X > 500) = 1 - \Phi\left(\frac{500 - 485}{30}\right) = 1 - \Phi(0.5) = 0.30854$, i.e., 30.854%.
 \diamond

11. **Mathemusical Bonus:** Suppose that a , k , and e are all nonzero. Use Beatles lyrics to prove that $m = t$.

Solution: According to “The End”, the Beatles state that “The love you take is equal to the love you make.” Canceling all of the similar terms and dividing by ake , we obtain the desired result. \diamond