ISyE 6739 — Summer 2009

Homework #6 (Covers Modules 19–26) — Solutions

All of the following problems are from Hines, et al.

4–18. m.g.f.

If $X_2 = A + BX_1$, show that $M_{X_2}(t) = e^{At} M_{X_1}(Bt)$.

Solution:

$$\begin{aligned} M_{X_2}(t) &= \mathsf{E}(e^{tX_2}) = \mathsf{E}(e^{t(A+BX_1)}) = \mathsf{E}(e^{At} \cdot e^{BtX_1}) \\ &= e^{At} \cdot \mathsf{E}(e^{BtX_1}) = e^{At} \cdot M_{X_1}(Bt). \quad \diamondsuit \end{aligned}$$

5–2. binomial.

Six independent trips to the moon are planned, each of which has estimated success probability 0.95. What's the probability that at least 5 will be successful?

Solution:

$$\begin{aligned} \mathsf{Pr}(X \geq 5) &= \sum_{x=5}^{6} \binom{6}{x} (0.95)^x (0.05)^{6-x} \\ &= 6(0.95)^5(0.05) + (0.95)^6 \\ &= 0.9672. \quad \diamondsuit \end{aligned}$$

5–6. binomial m.g.f.

Find the mean and variance of the binomial using the m.g.f.

Solution:

$$\begin{aligned} M_X(t) &= \mathsf{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= (pe^t + q)^n, \quad \text{where } q = 1-p. \end{aligned}$$

$$\mathsf{E}[X] = M_X'(0) = [n(pe^t + q)^{n-1} pe^t]|_{t=0} = np.$$

$$\begin{aligned} \mathsf{E}[X^2] &= M_X''(0) = np[e^t(n-1)(pe^t+q)^{n-2}(pe^t)+(pe^t+q)^{n-1}e^t]|_{t=0} \\ &= (np)^2 - np^2 + np. \end{aligned}$$

$$\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq. \quad \diamondsuit$$

<u>5–9. geometric.</u>

The probability of a successful firing of a cruise missile is 0.95. Assuming independent tests, what's the prob that the first failure occurs with the fifth missile?

Solution: $\mathsf{Pr}(X=5) = (0.95)^4(0.05) = 0.0407. \quad \diamondsuit$

<u>5–30. Poisson.</u>

Phone calls arrive at a switchboard according to a Pois(10/hour) process. The current system can handle up to 20 calls in an hour without becoming overloaded. What's the probability of an overload in the next hour?

Solution:

$$\begin{aligned} \mathsf{Pr}(X>20) = \mathsf{Pr}(X \geq 21) &= \sum_{x=21}^{\infty} \frac{e^{-10}(10)^x}{x!} \\ &= 1 - \mathsf{Pr}(X \leq 20) = 1 - \sum_{x=0}^{20} \frac{e^{-10}(10)^x}{x!} \\ &= 0.002. \quad \diamondsuit \end{aligned}$$

<u>6–13. exponential.</u>

The time to failure of a TV is exponential with a mean of 3 years. A company offers insurance for the first year of usage. On what percentage of policies will the company have to pay claims?

Solution: Let X = Life Length.

$$\mathsf{E}(X) = \frac{1}{\lambda} = 3 \quad \Rightarrow \quad \lambda = \frac{1}{3},$$

so

$$\mathsf{Pr}(X<1) = 1 - e^{-1/3} = 0.283.$$

Thus, 28.3% of policies result in a claim. $\diamondsuit$

6–16. exponential.

A transistor has an exponential time-to-failure distribution with a mean-time-to-failure of 20,000 hours. Suppose that the transistor has already lasted 20,000 hours. What's the probability that it fails by 30,000 hours?

Solution:

$$\mathsf{Pr}(X > x + s | X > x) \; = \; \mathsf{Pr}(X > s) \; = \; \mathsf{Pr}(X > 10000) \; = \; e^{-10000/20000} \; = \; 0.6064,$$

so $\mathsf{Pr}(X < 30000 | X > 20000) \; = \; 0.3936.$ $\diamondsuit$

7–1(a)–(e). normal.

Suppose Z is standard normal. Find

(a) $\mathsf{Pr}(0 < Z < 2)$.

(b) $\mathsf{Pr}(-1 < Z < 1)$.

(c) $\mathsf{Pr}(Z < 1.65)$.

(d) $\mathsf{Pr}(Z > -1.96)$.

(e) $\mathsf{Pr}(|Z| > 1.5)$.

Solution:

(a) $\mathsf{Pr}(0 \le Z \le 2) = \Phi(2) - \Phi(0) = 0.97725 - 0.5 = 0.47725$.

(b) $\mathsf{Pr}(-1 \le Z \le 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68268$.

(c) $\mathsf{Pr}(Z \le 1.65) = \Phi(1.65) = 0.95053$.

(d) $\mathsf{Pr}(Z \ge -1.96) = \Phi(1.96) = 0.9750$.

(e) $\mathsf{Pr}(|Z| \ge 1.5) = 2[1 - \Phi(1.5)] = 0.1336$. $\diamondsuit$

7–3(a). normal.

Find c such that $\Phi(c) = 0.94062$.

Solution: From Table II of the Appendix, $c = 1.56$. $\diamondsuit$

<u>7–5(a). normal.</u>

If $X \sim N(80, 10^2)$, find $\mathsf{Pr}(X < 100)$.

Solution: $\mathsf{Pr}(X \le 100) = \Phi\Big(\dfrac{100 - 80}{10}\Big) = \Phi(2) = 0.97725$. $\diamondsuit$

<u>7–7. normal.</u>

A manager requires job applicants to take a test and score a 500. The test scores are normally distributed with a mean of 485 and standard deviation of 30. What percent of applicants pass?

Solution:
$\mathsf{Pr}(X > 500) = 1 - \Phi\Big(\dfrac{500 - 485}{30}\Big) = 1 - \Phi(0.5) = 0.30854$, i.e., 30.854%. $\diamondsuit$