ISyE 6739 – Summer 2009

Homework #5 (Covers Modules 16–18) — Solutions

1. (Hines, et al., 4–1). A refrigerator manufacturer subjects his finished products to a final inspection. Of interest are two categories of defects: scratches or flaws in the porcelain finish, and mechanical defects. The number of each type of defects is a random variable. The results of inspecting 50 refrigerators are shown in the following joint p.m.f. table, where X represents the occurrence of finish defects and Y represents the occurrence of mechanical defects.

$Y \backslash X$	0	1	2	3	4	5
0	11/50	4/50	2/50	1/50	1/50	1/50
1	8/50	3/50	2/50	1/50	1/50	
2	4/50	3/50	2/50	1/50		
3	3/50	1/50				
4	1/50					

(a) Find the marginal distributions of X and Y.

Solution: Let's re-write the table, this time including the marginals.

$Y \backslash X$	0	1	2	3	4	5	$f_Y(y)$
0	11/50	4/50	2/50	1/50	1/50	1/50	20/50
1	8/50	3/50	2/50	1/50	1/50		15/50
2	4/50	3/50	2/50	1/50			10/50
3	3/50	1/50					4/50
4	1/50						1/50
$f_X(x)$	27/50	11/50	6/50	3/50	2/50	1/50	

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(b) Find the marginal distribution of mechanical defects, given that there are no finish defects.

Solution:

$$f(y|X=0) \;=\; \frac{f(0,y)}{f_X(0)} \;=\; \frac{f(0,y)}{27/50} \;=\; \begin{cases} 11/27 & \text{if } y=0 \\ 8/27 & \text{if } y=1 \\ 4/27 & \text{if } y=2 \\ 3/27 & \text{if } y=3 \\ 1/27 & \text{if } y=4 \\ 0 & \text{otherwise} \end{cases}$$

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(c) Find the marginal distribution of finish defects, given that there are no mechanical defects.

Solution:

$$f(x|Y=0) \;=\; \frac{f(x,0)}{f_Y(0)} \;=\; \frac{f(x,0)}{20/50} \;=\; \begin{cases} 11/20 & \text{if } x=0 \\ 4/20 & \text{if } x=1 \\ 2/20 & \text{if } x=2 \\ 1/20 & \text{if } x=3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

$\diamondsuit$

2. (Hines, et al., 4–4). Consider a situation in which the surface tension and acidity of a chemical product are measured. These variables are coded such that surface tension is measured on a scale $0 \le X \le 2$, and acidity is measured on a scale $2 \le Y \le 4$. The probability density function of (X,Y) is

$$f(x,y) \;=\; \begin{cases} k(6-x-y) & \text{if } 0 \le x \le 2,\ 2 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the appropriate value of k.

Solution: Setting

$$\int_2^4 \int_0^2 k(6-x-y)\,dx\,dy \;=\; 1,$$

we find that $k = 1/8$. $\diamondsuit$

(b) Calculate the probability that $X < 1,\ Y < 3$.

Solution:

$$\mathsf{Pr}(X<1, Y<3) \;=\; \int_2^3 \int_0^1 (1/8)(6-x-y)\,dx\,dy \;=\; 3/8. \quad \diamondsuit$$

(c) Calculate the probability that $X + Y \le 4$.

Solution: This is a little tough; pay attention to the limits.

$$\mathsf{Pr}(X+Y \le 4) \;=\; \int_2^4 \int_0^{4-y} (1/8)(6-x-y)\,dx\,dy \;=\; 2/3. \quad \diamondsuit$$

(d) Find the probability that $X < 1.5$.

Solution:

$$f_X(x) \;=\; \int_{\Re} f(x,y)\,dy \;=\; \int_2^4 (1/8)(6-x-y)\,dy \;=\; \frac{3-x}{4}, \quad 0 < x < 2.$$

This implies that

$$\mathsf{Pr}(X < 1.5) \;=\; \int_0^{1.5} \frac{3-x}{4}\,dx \;=\; 0.844. \quad \diamondsuit$$

(e) Find the marginal densities of both X and Y.

Solution: $f_X(x)$ is given above. Similarly,

$$f_Y(y) \;=\; \int_{\Re} f(x,y)\,dx \;=\; \int_0^2 (1/8)(6-x-y)\,dx \;=\; \frac{5-y}{4}, \quad 2<y<4. \quad \diamondsuit$$

3. Suppose that $f(x,y) = cxy^2$ for $0 < x < y^2 < 1$ and $0 < y < 1$.

(a) Find c.

Solution:

$$1 \;=\; \int\int_{\Re^2} f(x,y)\,dx\,dy \;=\; \int_0^1 \int_0^{y^2} cxy^2\,dx\,dy \;=\; c/14.$$

This immediately implies that $c = 14$. $\diamondsuit$

(b) Find the marginal p.d.f. of X, $f_X(x)$.

Solution:

$$f_X(x) \;=\; \int_{\Re} f(x,y)\,dy \;=\; \int_{\sqrt{x}}^1 14xy^2\,dy \;=\; \frac{14}{3}(x - x^{5/2}), \quad 0<x<1. \quad \diamondsuit$$

(c) Find the marginal p.d.f. of Y, $f_Y(y)$.

Solution:

$$f_Y(y) \;=\; \int_{\Re} f(x,y)\,dx \;=\; \int_0^{y^2} 14xy^2\,dx \;=\; 7y^6, \quad 0<y<1. \quad \diamondsuit$$

(d) Find $\mathsf{E}[X]$.

Solution:

$$\mathsf{E}[X] \;=\; \int_{\Re} x f_X(x)\,dx \;=\; \int_0^1 \frac{14}{3}(x^2 - x^{7/2})\,dx \;=\; \frac{14}{27}. \quad \diamondsuit$$

(e) Find $\mathsf{E}[Y]$.

Solution:

$$\mathsf{E}[Y] \;=\; \int_{\Re} y f_Y(y)\,dy \;=\; \int_0^1 7y^7\,dy \;=\; \frac{7}{8}. \quad \diamondsuit$$

(f) Find the conditional p.d.f. of X given $Y = y$, $f(x|y)$.

Solution:

$$f(x|y) \;=\; \frac{f(x,y)}{f_Y(y)} \;=\; \frac{2x}{y^4},\; 0 < x < y^2 < 1. \quad \diamondsuit$$

(g) Find the conditional expectation, $E[X|y]$.

Solution:

$$\mathsf{E}[X|y] \;=\; \int_{\Re} x f(x|y)\, dx \;=\; \int_0^{y^2} \frac{2x}{y^4}\, dx \;=\; \frac{2y^2}{3}. \quad \diamondsuit$$

(h) Find the "double" conditional expectation, $\mathsf{E}[\mathsf{E}[X|Y]]$.

Solution:

$$\mathsf{E}[\mathsf{E}[X|Y]] \;=\; \int_{\Re} \mathsf{E}[X|y] f_Y(y)\, dy \;=\; \int_0^1 \frac{2y^2}{3} 7y^6\, dy \;=\; \frac{14}{27}. \quad \diamondsuit$$

4. (Hines, et al., 4–8.) Consider the probability distribution of the discrete random vector (X, Y), where X represents the number of orders for aspirin in August in the neighborhood drugstore and Y represents the number of orders in September. The joint distribution is shown in the following table.

$Y\backslash X$	51	52	53	54	55
51	0.06	0.05	0.05	0.01	0.01
52	0.07	0.05	0.01	0.01	0.01
53	0.05	0.10	0.10	0.05	0.05
54	0.05	0.02	0.01	0.01	0.03
55	0.05	0.06	0.05	0.01	0.03

(a) Find the marginal distributions.

Solution: After we add up the usual stuff, we get the following marginals:

z	51	52	53	54	55
$f_X(z)$	0.28	0.28	0.22	0.09	0.13
$f_Y(z)$	0.18	0.15	0.35	0.12	0.20

$\diamondsuit$

(b) Find the expected sales in September, given that sales in August were either 51, 52, 53, 54, or 55, respectively.

Solution:

$$\mathsf{E}[Y|X=x] \;=\; \sum_y y f_{Y|X}(y|x) \;=\; \frac{1}{f_X(x)}\sum_y y f(x,y).$$

For example, we get

$$\mathsf{E}[Y|X=51] \;=\; \frac{1}{0.28}[(51)(0.06)+(52)(0.07)+\cdots+(55)(0.05)].$$

Similarly, we get the following table.

x	51	52	53	54	55
$\mathsf{E}[Y\|X=x]$	52.86	52.96	53	53	53.46

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5. (Hines, et al., 4–9). Assume that X and Y are coded scores of two intelligence tests, and the p.d.f. of (X,Y) is given by

$$f(x,y) \;=\; \begin{cases} 6x^2y & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of the score on test #2 given the score on test #1. Then find the expected value of the score on test #1 given the score on test #2.

Hints: After the usual algebra and calculus, you get

$$\mathsf{E}[X|Y=y] \;=\; 3/4$$

and

$$\mathsf{E}[Y|X=x] \;=\; 2/3.$$

Note that the answers don't depend on y or x, respectively! This is because X and Y are *independent* (Why?)

6. (Hines, et al., 4–31). Given the following joint p.d.f.'s, determine whether or not X and Y are independent.

(a) $g(x,y) = 4xye^{-(x^2+y^2)}$, $x > 0$, $y > 0$.

Solution: Since (i) there are no funny limits and (ii) you can factor $g(x,y) = (4xe^{-x^2})(ye^{-y^2})$, we see that X and Y are independent. ♢

(b) $f(x,y) = 3x^2y^{-3}$, $0 < x < y < 1$.
Solution: Funny limits imply *not* independent. $\diamondsuit$

(c) $f(x,y) = 6(1+x+y)^{-4}$, $x > 0$, $y > 0$.
Solution: Can't factor $f(x,y) = g(x)h(y)$ implies *not* independent. $\diamondsuit$

7. (Hines, et al., 4–19). Let X and Y have joint p.d.f. $f(x,y) = 2$, $0 < x < y < 1$. Find the correlation between X and Y.

Solution: I won't go through all of the tedious calculations, but here are the highlights.

$$f_X(x) \;=\; \int_x^1 2\,dy \;=\; 2(1-x), \quad 0 < x < 1$$

and

$$f_Y(y) \;=\; \int_0^y 2\,dx \;=\; 2y, \quad 0 < y < 1.$$

Then you get (in the usual way)

$$\mathsf{E}[X] \;=\; 1/3, \quad \mathsf{Var}(X) \;=\; 1/18, \quad \mathsf{E}[Y] \;=\; 2/3, \quad \mathsf{Var}(Y) \;=\; 1/18.$$

Further,

$$\mathsf{E}[XY] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy \;=\; \int_0^1\int_0^{x_2} 2xy\,dx\,dy \;=\; 1/4.$$

This finally implies that

$$\rho \;=\; \frac{\mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; 0.5. \quad \diamondsuit$$

8. (Hines, et al., 4–21). Consider the data in Problem 1 (Hines, et al., 4–1) above. Are X and Y independent? Find the correlation.

Hints: After the usual manipulations, get $\rho = -0.1355$.

9. Let $\mathsf{Var}(X) = \mathsf{Var}(Y) = 20$, $\mathsf{Var}(Z) = 30$, $\mathsf{Cov}(X,Y) = 2$, $\mathsf{Cov}(X,Z) = -3$, and $\mathsf{Cov}(Y,Z) = -4$. Find $\mathsf{Corr}(X,Z)$ and $\mathsf{Var}(X - 2Y + 5Z)$.

Solution:

$$\mathsf{Corr}(X,Z) \;=\; \frac{\mathsf{Cov}(X,Z)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Z)}} \;=\; -0.1225$$

and

$$\begin{aligned}\mathsf{Var}(X - 2Y + 5Z) \;&=\; \mathsf{Var}(X) + 4\mathsf{Var}(Y) + 25\mathsf{Var}(Z)\\ &\qquad -2\cdot 2\mathsf{Cov}(X,Y) + 2\cdot 5\mathsf{Cov}(X,Z) - 2\cdot 10\mathsf{Cov}(Y,Z)\\ &=\; 892. \quad \diamondsuit\end{aligned}$$