ISyE 6739 — Summer 2009

Homework #3 (Covers Modules 10–14) — Solutions

1. A die is rolled 5 times. Let X denote the number of times that you see a 3 or 6.

   (a) What's the distribution of X?
   Solution: $X \sim \text{Bin}(5, 1/3)$. $\diamondsuit$

   (b) Find $\mathsf{Pr}(X = 4)$.
   Solution:

   $$\mathsf{Pr}(X = 4) \;=\; \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{5-4} \;=\; 0.0412. \quad \diamondsuit$$

2. Suppose $X \sim \text{Pois}(2)$. Find $\mathsf{Pr}(X \le 3)$.

   Solution:

   $$\mathsf{Pr}(X \le 3) \;=\; \sum_{k=0}^{3} e^{-2}2^k/k! \;=\; 0.8571. \quad \diamondsuit$$

3. Suppose X has the following discrete distribution.

x	-1	0	2	3
$\mathsf{Pr}(X = x)$	0.2c	0.3	0.2	0.1

   (a) Find the value of c that will make the p.m.f. sum to 1.
   Solution: Note that

   $$1 \;=\; \sum_x \mathsf{Pr}(X = x) \;=\; 0.2c + 0.3 + 0.2 + 0.1.$$

   This implies that $c = 2$. $\diamondsuit$

   (b) Find the c.d.f. $F(x)$ for all x.
   Solution: We have

   $$F(x) \;=\; \mathsf{Pr}(X \le x) \;=\; \begin{cases} 0 & \text{if } x < -1 \\ 0.4 & \text{if } -1 \le x < 0 \\ 0.7 & \text{if } 0 \le x < 2 \\ 0.9 & \text{if } 2 \le x < 3 \\ 1.0 & \text{if } x \ge 3 \end{cases} \qquad \diamondsuit$$

   (c) Calculate $\mathsf{E}[X]$.
   Solution:

   $$\mathsf{E}[X] \;=\; \sum_x x\mathsf{Pr}(X = x) \;=\; 0.3. \quad \diamondsuit$$

(d) Calculate $\mathsf{Var}(X)$.
Solution:

$$\mathsf{E}[X^2] \;=\; \sum_x x^2 \mathsf{Pr}(X = x) \;=\; 2.1.$$

This implies that $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 2.01.$ $\diamondsuit$

(e) Calculate $\mathsf{Pr}(1 \le X \le 2)$.
Solution: Since X can only equal $-1, 0, 2, 3$, we have $\mathsf{Pr}(1 \le X \le 2) = \mathsf{Pr}(X = 2) = 0.2.$ $\diamondsuit$

4. Suppose that X is continuous with p.d.f. $f(x) = cx^2$, $0 \le x \le 1$.

(a) Find the value of c that will make the p.d.f. integrate to 1.
Solution: Note that

$$1 \;=\; \int_{\Re} f(x)\,dx \;=\; \int_0^1 cx^2\,dx \;=\; c/3.$$

This implies that $c = 3$. $\diamondsuit$

(b) Find the c.d.f. $F(x)$ for all x.
Solution:

$$F(x) \;=\; \mathsf{Pr}(X \le x) \;=\; \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \le x \le 1 \\ 1.0 & \text{if } x \ge 1 \end{cases} \qquad \diamondsuit$$

(c) Calculate $\mathsf{E}[X]$.
Solution:

$$\mathsf{E}[X] \;=\; \int_{\Re} x f(x)\,dx \;=\; \int_0^1 3x^3\,dx = 3/4. \qquad \diamondsuit$$

(d) Calculate $\mathsf{Var}(X)$.
Solution: Similarly,

$$\mathsf{E}[X^2] \;=\; \int_{\Re} x^2 f(x)\,dx \;=\; \int_0^1 3x^4\,dx \;=\; 3/5.$$

This implies that $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 3/80.$ $\diamondsuit$

(e) Calculate $\mathsf{Pr}(0 \le X \le 1/2)$.
Solution: $\mathsf{Pr}(0 \le X \le 1/2) = F(1/2) - F(0) = 1/8.$ $\diamondsuit$

5. Let $\mathsf{E}[X] = -4$, $\mathsf{Var}(X) = 5$, and $Z = -4X + 7$. Find $\mathsf{E}[3Z]$ and $\mathsf{Var}(3Z)$.

   Solution: We have

$$\mathsf{E}[3Z] \;=\; \mathsf{E}[-12X + 21] \;=\; -12\mathsf{E}[X] + 21 \;=\; 69. \quad \diamondsuit$$

   and

$$\mathsf{Var}(3Z) \;=\; \mathsf{Var}(-12X + 21) \;=\; 144\mathsf{Var}(X) \;=\; 720. \quad \diamondsuit$$

6. When a machine is adjusted properly, 50% of the items it produces are good and 50% are bad. However, the machine is *improperly* adjusted 10% of the time; in this case, 25% of the items it makes are good and 75% are bad. Suppose that 5 items produced by the machine are selected at random and inspected. If 4 of these items are good (and 1 is bad), what's the probability that the machine was adjusted properly at the time? **Hint:** Try Bayes Theorem using Binomial conditional probabilities.

   Solution: Let X be the number of good items (out of 5). Further, define the following events.

$$\begin{aligned} P &= \text{“machine is properly adjusted”, which implies } X \sim \text{Bin}(5, 1/2) \\ I &= \text{“machine is improperly adjusted”, which implies } X \sim \text{Bin}(5, 1/4) \end{aligned}$$

   Now Bayes implies that

$$\begin{aligned} \mathsf{Pr}(P|X=4) &= \frac{\mathsf{Pr}(X=4|P)\mathsf{Pr}(P)}{\mathsf{Pr}(X=4|P)\mathsf{Pr}(P) + \mathsf{Pr}(X=4|I)\mathsf{Pr}(I)} \\ &= \frac{\binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1(0.9)}{\binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1(0.9) + \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1(0.1)} \\ &= 0.9897. \quad \diamondsuit \end{aligned}$$

7. Suppose that $X \sim \text{Unif}(-1, 6)$. Compare the upper bound on the probability $\mathsf{Pr}(|X - \mu| \geq 1.5\sigma)$ obtained from Chebychev's inequality with the exact probability.

   Solution: By Chebychev, we get the following bound.

$$\mathsf{Pr}(|X - \mu| \geq 1.5\sigma) \;\leq\; \frac{1}{(1.5)^2} \;=\; 4/9. \quad \diamondsuit$$

   Now let's get the exact probability. First of all, recall that if $X \sim \text{U}(a, b)$, then $\mathsf{E}[X] = (a + b)/2$ and $\mathsf{Var}(X) = (a - b)^2/12$. Since $X \sim \text{U}(-1, 6)$, we have

$\mathsf{E}[X] = 5/2$ and $\sigma^2 = \mathsf{Var}(X) = 49/12$. Thus, we can make the following exact calculations.

$$\begin{aligned}
\mathsf{Pr}(|X-\mu| \geq 1.5\sigma) &= \mathsf{Pr}\left(\left|X - \frac{5}{2}\right| \geq 1.5 \cdot \sqrt{49/12}\right) \\
&= 1 - \mathsf{Pr}\left(\left|X - \frac{5}{2}\right| < 3.03\right) \\
&= 1 - \mathsf{Pr}\left(-3.03 < X - \frac{5}{2} < 3.03\right) \\
&= 1 - \mathsf{Pr}\left(-0.53 < X < 5.53\right) \\
&= 1 - \int_{-0.53}^{5.53} f(x)\, dx \\
&= 1 - \frac{1}{7}(5.53 + 0.53) \;=\; 0.1343. \qquad \diamondsuit
\end{aligned}$$