

ISyE 6739 – Summer 2009

Homework #2 (Covers Modules 6–9) — Solutions

1. A bridge hand contains 13 cards from a standard deck. Find the probability that a bridge hand will contain

(a) Exactly 2 aces.

Solution: $\frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}}$ \square

(b) At least 2 aces.

Solution: $\sum_{i=2}^4 \binom{4}{i} \binom{48}{13-i} / \binom{52}{13}$ \square

(c) 8 spades.

Solution: $\frac{\binom{13}{8}\binom{39}{5}}{\binom{52}{13}}$ \square

(d) 8 cards of the same suit.

Solution: $4 \frac{\binom{13}{8}\binom{39}{5}}{\binom{52}{13}}$ \square

2. A die is thrown 7 times. Find

(a) $\Pr(\text{'6' comes up at least once})$.

Solution: $1 - \Pr(\text{no 6's appear}) = 1 - (5/6)^7$ \square .

(b) $\Pr(\text{each face appears at least once})$.

Solution: Denote the six faces by A,B,C,D,E,F. Thus, we need to find the number of tosses of the form A,A,B,C,D,E,F. We then see that

i. The # ways to choose A is 6.

ii. The # ways to place A is $\binom{7}{2}$.

iii. The # ways to permute B,C,D,E,F is 5!.

iv. The # ways to toss the die 7 times is 6^7 .

Thus,

$$\Pr(\text{each face appears at least once}) = 6 \cdot \binom{7}{2} \cdot 5!/6^7.$$

3. Write a computer program in your favorite language to calculate combinations. Demonstrate your program on $C_{100,50}$.

Solution: Don't try to calculate big numbers like $100!$ directly. Make use of fractions to keep the sizes reasonable. Try something like

$$\binom{100}{50} = \frac{100!}{50!50!} = \frac{100 \cdot 99 \cdots 51}{50 \cdot 49 \cdots 1},$$

which should help you get around overflow problems. By the way, it turns out that

$$\binom{100}{50} = 100891344545564193334812497256. \quad \square$$

4. Twenty items (12 bad, 8 good) are inspected. If the items are chosen at random, what's the probability that

(a) The first two are bad?

Solution: $\frac{12}{20} \cdot \frac{11}{19} = \frac{33}{95} \quad \square$

(b) The first two are good?

Solution: $\frac{8}{20} \cdot \frac{7}{19} = \frac{14}{95} \quad \square$

(c) One of each in the first two?

Solution: $1 - \frac{33}{95} - \frac{14}{95} = \frac{48}{95} \quad \square$

5. A box contains 4 bad and 6 good tubes. Two are drawn out. One is tested and found to be good. What's the probability that the other is good?

Solution: $\Pr(\text{second is good} | \text{first is good}) = 5/9 \quad \square$

6. In a class there are 4 freshman boys, 6 freshman girls, and 6 sophomore boys. How many sophomore girls must be present if gender and class are to be independent when a student is selected at random?

Solution: Let x be the number of soph girls. In order for gender and class to be indep, you need to have

$$\Pr(B \cap F) = \Pr(B)\Pr(F)$$

$$\Pr(G \cap F) = \Pr(G)\Pr(F)$$

$$\Pr(B \cap S) = \Pr(B)\Pr(S)$$

$$\Pr(G \cap S) = \Pr(G)\Pr(S)$$

In other words,

$$\begin{aligned} \frac{4}{16+x} &= \frac{10}{16+x} \cdot \frac{10}{16+x} \\ \frac{6}{16+x} &= \frac{6+x}{16+x} \cdot \frac{10}{16+x} \\ \frac{6}{16+x} &= \frac{10}{16+x} \cdot \frac{6+x}{16+x} \\ \frac{x}{16+x} &= \frac{6+x}{16+x} \cdot \frac{6+x}{16+x} \end{aligned}$$

Solving any of these equations for x , we obtain $x = 9$. \square .

7. If A and B are independent, $\Pr(A) = 0.4$, and $\Pr(A \cup B) = 0.6$, find $\Pr(B)$.

Solution: Since A, B are indep, we have

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B).$$

This implies

$$\Pr(B) = \frac{\Pr(A \cup B) - \Pr(A)}{1 - \Pr(A)} = \frac{0.6 - 0.4}{1 - 0.4} = 1/3. \quad \square$$

8. Suppose $\Pr(A) = 0.4$, $\Pr(A \cup B) = 0.7$, and $\Pr(B) = x$.

- (a) For what choice of x are A and B disjoint?

Solution: Since A, B are disjct, we have

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \Pr(A) + \Pr(B).$$

This implies $\Pr(B) = x = 0.3$. \square

- (b) Independent?

Solution: As in Problem 7, we get $x = 0.5$. \square

9. Prove: If $\Pr(A|B) > \Pr(A)$, then $\Pr(B|A) > \Pr(B)$.

Solution:

$$\begin{aligned} \Pr(A|B) &> \Pr(A) \\ \Rightarrow \Pr(A \cap B)/\Pr(B) &> \Pr(A) \\ \Rightarrow \Pr(A \cap B) &> \Pr(A)\Pr(B) \\ \Rightarrow \Pr(A \cap B)/\Pr(A) &> \Pr(B) \\ \Rightarrow \Pr(B|A) &> \Pr(B) \quad \square. \end{aligned}$$

10. The probability of scoring a basket is p . Joe shoots first. If he misses, Fred gets to shoot. They shoot the ball back and forth until somebody scores. What's the probability that Joe wins? Graph this as a function of p .

Solution:

$$\begin{aligned}
 \Pr(\text{Joe wins}) &= \Pr(\text{Joe wins on 1st roll}) + \Pr(\text{Joe wins on 3rd roll}) + \dots \\
 &= \Pr(\text{Joe hits on 1st roll}) + \Pr(\text{Joe misses, Fred misses, Joe hits}) + \dots \\
 &= p + (1-p)(1-p)p + (1-p)(1-p)(1-p)(1-p)p + \dots \\
 &= p \sum_{i=0}^{\infty} (1-p)^{2i} \\
 &= \frac{p}{1 - (1-p)^2} \\
 &= \frac{1}{2-p}. \quad \square
 \end{aligned}$$

11. Consider two boxes. The first box contains one black marble and one white marble. The second box contains two blacks and one white. A box is selected at random and a marble is drawn at random from the selected box.

- (a) Find $\Pr(\text{the marble is black})$.

Solution: Let B, W denote black and white respectively, and let I, II denote the selection of Box 1 and Box 2. We use the law of total probability to find

$$\begin{aligned}
 \Pr(B) &= \Pr(B|I)\Pr(I) + \Pr(B|II)\Pr(II) \\
 &= \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} \\
 &= 7/12. \quad \square
 \end{aligned}$$

- (b) What is the probability that the marble was selected from the first box given that the marble is white?

Solution: We use Bayes Theorem to find

$$\begin{aligned}
 \Pr(I|W) &= \frac{\Pr(W|I)\Pr(I)}{\Pr(W|I)\Pr(I) + \Pr(W|II)\Pr(II)} \\
 &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}} \\
 &= 3/5. \quad \square
 \end{aligned}$$

12. A gambler has in his pocket a fair coin and a two-headed coin.

- (a) He selects one at random, and when he flips it, it shows heads. What's the probability that the coin is fair?

Solution: Let F, U denote fair and unfair respectively. We use Bayes to find

$$\begin{aligned}\Pr(F|H) &= \frac{\Pr(H|F)\Pr(F)}{\Pr(H|F)\Pr(F) + \Pr(H|U)\Pr(U)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} \\ &= 1/3. \quad \square\end{aligned}$$

- (b) He flips the same coin and it again shows heads. Same question.

Solution: As above,

$$\begin{aligned}\Pr(F|HH) &= \frac{\Pr(HH|F)\Pr(F)}{\Pr(HH|F)\Pr(F) + \Pr(HH|U)\Pr(U)} \\ &= \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} \\ &= 1/5. \quad \square\end{aligned}$$

- (c) He flips it a third time and it shows tails. Same question.

Solution: $\Pr(F|HHT) = 1$ (trivially!) \square

13. (A tough problem!) 3 prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other 2 are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses by arguing that if A knew, A's probability of being executed would rise from $1/3$ to $1/2$ (i.e., there would only be 2 prisoners left). Whose reasoning is correct and why?

Solution: Label the prisoners A,B,C. Without loss of generality, suppose the jailer says "It's not B", which we'll denote by NB. What's the conditional probability that A dies?

By Bayes, we have

$$\begin{aligned}\Pr(\text{A dies} | \text{Jailer says "It's not B"}) &= \frac{\Pr(\text{Jailer says "It's not B"} | \text{A dies})\Pr(\text{A dies})}{\Pr(\text{NB} | \text{A dies})\Pr(\text{A dies}) + \Pr(\text{NB} | \text{B dies})\Pr(\text{B dies}) + \Pr(\text{NB} | \text{C dies})\Pr(\text{C dies})} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = 1/3.\end{aligned}$$

Thus, A is correct! \square