ISyE 6739 – Summer 2009

Homework #1 (Covers Modules 1–5) — Solutions

1. Suppose $U = [0,2]$, $A = [0.5,1]$, and $B = [0.25,1.5]$. What are $\overline{A \cup B}$, $A \cup \overline{B}$, $\overline{A \cap B}$, and $\overline{A} \cap B$?

   Solution: Note that $A \subseteq B$. Then

   $\overline{A \cup B} = [0, \frac{1}{4}) \cup (\frac{3}{2}, 2]$

   $A \cup \overline{B} = [0, \frac{1}{4}) \cup [\frac{1}{2}, 1] \cup (\frac{3}{2}, 2]$

   $\overline{A \cap B} = [0, \frac{1}{2}) \cup (1, 2]$

   $\overline{A} \cap B = [\frac{1}{4}, \frac{1}{2}) \cup (1, \frac{3}{2}]$ $\Box$

2. Prove DeMorgan's Laws. You can use Venn diagrams or argue mathematically.

   Solution: Let's prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

   Proof:

$$\begin{aligned} x \in \overline{A \cup B} \quad & \text{iff} \quad x \notin A \cup B \\ & \text{iff} \quad x \notin A \text{ and } x \notin B \\ & \text{iff} \quad x \in \overline{A} \text{ and } x \in \overline{B} \\ & \text{iff} \quad x \in \overline{A} \cap \overline{B} \quad \Box \end{aligned}$$

   Now let's prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

   Proof:

$$\begin{aligned} x \in \overline{A \cap B} \quad & \text{iff} \quad x \notin A \cap B \\ & \text{iff} \quad x \notin A \text{ or } x \notin B \\ & \text{iff} \quad x \in \overline{A} \text{ or } x \in \overline{B} \\ & \text{iff} \quad x \in \overline{A} \cup \overline{B} \quad \Box \end{aligned}$$

3. A box contains 3 marbles (one red, one green, and one blue).

   (a) Consider an experiment that consists of taking one marble from the box, then replacing it in the box, and then drawing a second marble from the box. What is the sample space?

   Solution: $S = \{RR, RG, RB, GR, GG, GB, BR, BG, BB\}$ $\Box$.

   (b) Repeat the above when the second marble is drawn *without* replacing the first marble.

   Solution: $S = \{RG, RB, GR, GB, BR, BG, \}$ $\Box$.

4. Let E, F, and G be 3 events. Suppose that $U = E \cup F \cup G$. Find expressions for the following events:

   (a) only F occurs.
   Solution: $\overline{E} \cap F \cap \overline{G}$ $(= \overline{E} \cap \overline{G},$ since $U = E \cup F \cup G)$ □

   (b) both E and F occur, but not G.
   Solution: $E \cap F \cap \overline{G}$ □

   (c) at least one event occurs.
   Solution: $E \cup F \cup G$ $(= U)$ □

   (d) at least two events occur.
   Solution: $(E \cap F) \cup (E \cap G) \cup (F \cap G)$ □

   (e) all three events occur.
   Solution: $E \cap F \cap G$ □

   (f) none occur.
   Solution: $\overline{E \cup F \cup G}$ $(= \emptyset)$, since "none" is the opposite of "at least one" □

   (g) at most one occurs.
   Solution:

$$\begin{aligned}
\text{"at most one"} &= \text{"none"} \cup \text{"exactly one"} \\
&= (\overline{E \cup F \cup G}) \cup [(E \cap \overline{F} \cap \overline{G}) \cup (\overline{E} \cap F \cap \overline{G}) \cup (\overline{E} \cap \overline{F} \cap G)] \\
&= (\overline{F} \cap \overline{G}) \cup (\overline{E} \cap \overline{G}) \cup (\overline{E} \cap \overline{F}) \quad (\text{since } U = E \cup F \cup G) \ \square
\end{aligned}$$

   (h) at most two occur.
   Solution: $\overline{E \cap F \cap G}$, since "at most 2" is the opposite of "all 3" □

5. How many 4-letter words can be formed from the alphabet if we require

   (a) The 2nd letter to be a vowel (a, e, i, o, u)?
   Solution: $26 \cdot 5 \cdot 26 \cdot 26$ □

   (b) Exactly one vowel?
   Solution:

$$\text{(Place vowel)}\cdot\text{(choose vowel)}\cdot\text{(choose 3 consonants)} = \binom{4}{1} 5 \cdot (21)^3 \quad \square$$

(c) At least one vowel?

Solution:

$$\text{(Total \# words)} - \text{(Words with 0 vowels)} \;=\; (26)^4 - (21)^4 \quad \Box$$

6. As if you have nothing better to do, toss a die 600 times. (I suppose you can do this in Excel.) How many times do each of the numbers come up? Approximately how many would you expect?

   Solution: You'd expect about 100 occurrences of each number (but results may vary). $\Box$

7. As if you still have nothing better to do, toss two dice 5000 times. (OK, I think it's time to rev up Excel.) How many times do each of the possible sums come up? Approximately how many would you expect?

   Solution: You'd expect about occurrences to come up in the following proportions.

sum	2	3	4	5	6	7	8	9	10	11	12
prob	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$\Box$

8. (Bonus) Three dice are tossed. What is the probability that the same number appears on exactly two of the three dice?

   Solution: There are actually a buncha ways to do this problem. For instance, let X equal the number of times a 1 appears on the three tosses. Then, as we will have learned by now, $X \sim \text{Bin}(3, 1/6)$, so that

$$\mathsf{Pr}(X = 2) \;=\; \binom{3}{2} (\frac{1}{6})^2 (\frac{5}{6}).$$

Thus, the prob that *some* number appears twice is

$$6 \cdot \mathsf{Pr}(X = 2) \;=\; 6 \cdot \binom{3}{2} (\frac{1}{6})^2 (\frac{5}{6}) \;=\; 0.417 \quad \Box.$$