39. Other Hypothesis Tests

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Normal Variance Test

Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Nor}(\mu, \sigma^2)$, where μ and σ^2 are *unknown*.

Two-sided test (you specify hypothesized σ_0^2):

$$H_0 : \sigma^2 = \sigma_0^2$$
$$H_1 : \sigma^2 \neq \sigma_0^2$$

We'll use the test statistic

$$\chi_0^2 \equiv \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1) \quad \text{(if } H_0 \text{ true)},$$

where

$$S^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \frac{\sigma^2\chi^2(n-1)}{n-1}.$$

Thus, we reject H_0 iff

$$\chi_0^2 < \chi^2_{1-\alpha/2,n-1} \quad \text{or} \quad \chi_0^2 > \chi^2_{\alpha/2,n-1}.$$

One-Sided Tests:

$$H_0 : \sigma^2 \le \sigma_0^2 \quad \text{vs.} \quad H_1 : \sigma^2 > \sigma_0^2$$

$$\text{Reject } H_0 \quad \text{iff} \quad \chi_0^2 \ > \ \chi^2_{\alpha,n-1}.$$

$$H_0 : \sigma^2 \ge \sigma_0^2 \quad \text{vs.} \quad H_1 : \sigma^2 < \sigma_0^2$$

$$\text{Reject } H_0 \quad \text{iff} \quad \chi_0^2 \ < \ \chi^2_{1-\alpha,n-1}.$$

Example: Suppose we want to test at level 0.05 whether or not the variance of a certain process is ≤ 0.02.

$$H_0 : \sigma^2 \leq 0.02$$
$$H_1 : \sigma^2 > 0.02$$

If the sample variance is “too high”, we’ll reject H_0.

Suppose we have $n = 20$, $\bar{X} = 125.12$, and $S^2 = 0.00225$.

Then the test stat $\chi_0^2 = (n-1)S^2/\sigma_0^2 = 21.375$ (and isn't explicitly dependent on $\bar{X}$).

Further, $\chi^2_{\alpha,n-1} = \chi^2_{.05,19} = 30.14$.

So we *fail to reject* H_0.

Two-Sample Test for Equal Variances

Do the two populations have the same variance?

$$X_1, X_2, \ldots, X_{n_x} \overset{\text{iid}}{\sim} \text{Nor}(\mu_x, \sigma_x^2) \quad \text{and}$$

$$Y_1, Y_2, \ldots, Y_{n_y} \overset{\text{iid}}{\sim} \text{Nor}(\mu_y, \sigma_y^2).$$

All X's and Y's are independent.

Two-sided test: $H_0 : \sigma_x^2 = \sigma_y^2$ vs. $H_1 : \sigma_x^2 \neq \sigma_y^2$

We'll use the test statistic

$$F_0 \equiv \frac{S_x^2}{S_y^2} \sim F(n_x - 1, n_y - 1) \quad \text{(if } H_0 \text{ true)},$$

where S_x^2 and S_y^2 are the two sample variances.

Thus, we reject H_0 iff

$$F_0 < F_{1-\alpha/2, n_x-1, n_y-1} \quad \text{or} \quad F_0 > F_{\alpha/2, n_x-1, n_y-1}.$$

iff

$$F_0 < \frac{1}{F_{\alpha/2, n_y-1, n_x-1}} \quad \text{or} \quad F_0 > F_{\alpha/2, n_x-1, n_y-1}.$$

One-Sided Tests:

$$H_0 : \sigma_x^2 \leq \sigma_y^2 \quad \text{vs.} \quad H_1 : \sigma_x^2 > \sigma_y^2$$

Reject H_0 iff $F_0 > F_{\alpha, n_x - 1, n_y - 1}$.

$$H_0 : \sigma_x^2 \geq \sigma_y^2 \quad \text{vs.} \quad H_1 : \sigma_x^2 < \sigma_y^2$$

Reject H_0 iff $F_0 < F_{1-\alpha, n_x - 1, n_y - 1} = \frac{1}{F_{\alpha, n_y - 1, n_x - 1}}$.

Example: Suppose we want to test at level 0.05 whether or not two processes have the same variance.

$$H_0 : \sigma_x^2 = \sigma_y^2$$
$$H_1 : \sigma_x^2 \neq \sigma_y^2$$

If the ratio of the sample variances is “too high” or “too low”, we'll reject H_0.

Suppose we have the following data:

$$n_x = n_y = 8, \; S_x^2 = 3.89, \; S_y^2 = 4.02.$$

Then $F_0 = S_x^2/S_y^2 = 0.968$,

$F_{1-\alpha/2,n_x-1,n_y-1} = 1/F_{.025,7,7} = 0.20$, and

$F_{\alpha/2,n_x-1,n_y-1} = F_{.025,7,7} = 4.99$.

Thus, we *fail to reject* H_0.

Bernoulli Proportion Test

Suppose that $X_1, X_2, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Bern}(p)$.

We're interested in testing hypotheses about the success parameter p.

Two-sided test (you specify hypothesized p_0):

$$H_0 : p = p_0$$
$$H_1 : p \neq p_0$$

Let $Y = \sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$

We'll use the test statistic

$$Z_0 \equiv \frac{Y - np_0}{\sqrt{np_0(1-p_0)}}.$$

If H_0 is true, the central limit theorem implies that

$$Z_0 \approx \text{Nor}(0, 1).$$

Thus, we reject H_0 iff $|Z_0| > z_{\alpha/2}$.

Tips: In order for the CLT to work, you need n large (say at least 30), and p not too close to 0 or 1.

Remark: If n isn't very big, you may have to use Binomial tables (instead of the normal approxn). This gets a little tedious, and I won't go into it here!

One-Sided Tests:

$$H_0 : p \leq p_0 \quad \text{vs.} \quad H_1 : p > p_0$$

$$\text{Reject } H_0 \quad \text{iff} \quad Z_0 > z_\alpha.$$

$$H_0 : p \geq p_0 \quad \text{vs.} \quad H_1 : p < p_0$$

$$\text{Reject } H_0 \quad \text{iff} \quad Z_0 < -z_\alpha.$$

Example: In 200 samples of a certain semiconductor, there were only 4 defectives. We're interested in proving "beyond a shadow of a doubt" that the probability of a defective is less than 0.06. Let's conduct the test at level 0.05.

$$H_0 : p \geq 0.06$$
$$H_1 : p < 0.06$$

(Since p is close to 0, we really did need to take a lot of observations — 200 in this case — in order for the CLT to work.)

We have $n = 200$, $Y = 4$ defectives, and $p_0 = 0.06$.

The test stat is

$$Z_0 = \frac{Y - np_0}{\sqrt{np_0(1 - p_0)}} = -2.357.$$

Since $-z_\alpha = -1.645$, we *reject* H_0.

I.e., it looks like p really is < 0.06.

Sample-Size Selection

Can we design a two-sided test $H_0 : p = p_0$ vs. $H_1 : p \neq p_0$ such that

$$\Pr(\text{Type I error}) \;=\; \alpha \quad \text{and}$$

$$\Pr(\text{Type II error} \,|\, p \neq p_0) \;=\; \beta?$$

I.e., can we specify the sample size for a two-sided test that will work when we require a Type I error bound α, and a Type II prob β? (The sample size will end up being a function of p as well. Read on. . .)

It can be shown (next pg) that the α and β design req'ts can be achieved by taking a sample of size

$$n \approx \left[\frac{z_{\alpha/2}\sqrt{p_0 q_0} + z_\beta \sqrt{pq}}{p - p_0}\right]^2,$$

where, to save space, we let $q \equiv 1 - p$ and $q_0 \equiv 1 - p_0$.

Notice that n is a function of the unknown p. In practice, we'll choose some $p = p_1$ and ask "How many obsns should I take if p happens to equal p_1 instead of p_0"? Thus, we guard against the scenario in which p actually equals p_1.

Proof:

$$
\begin{aligned}
\beta &= \text{Pr(Type II error)} \\
&= \Pr(\text{Fail to Reject } H_0 \,|\, H_0 \text{ false }) \\
&= \Pr(|Z_0| \le z_{\alpha/2} \,|\, p \ne p_0) \\
&= \Pr(-z_{\alpha/2} \le Z_0 \le z_{\alpha/2} \,|\, p \ne p_0) \\
&= \Pr\left(-z_{\alpha/2} \le \frac{Y - np_0}{\sqrt{np_0(1-p_0)}} \le z_{\alpha/2} \,\middle|\, p \ne p_0\right) \\
&= \Pr\left(-z_{\alpha/2}\sqrt{\frac{p_0 q_0}{pq}} \le \frac{Y - np_0}{\sqrt{npq}} \le z_{\alpha/2}\sqrt{\frac{p_0 q_0}{pq}} \,\middle|\, p \ne p_0\right)
\end{aligned}
$$

Continuing, we have

$$\begin{aligned}\beta &= \mathsf{Pr}\left(-c \le \frac{Y - np_0}{\sqrt{npq}} \le c \,\middle|\, p \ne p_0\right) \\ &= \mathsf{Pr}\left(-c \le \frac{Y - np}{\sqrt{npq}} + \frac{n(p - p_0)}{\sqrt{npq}} \le c \,\middle|\, p \ne p_0\right),\end{aligned}$$

where

$$c \equiv z_{\alpha/2}\sqrt{\frac{p_0 q_0}{pq}}.$$

Now notice that (since p is the true success prob),

$$Z \equiv \frac{Y - np}{\sqrt{npq}} \sim \mathsf{Nor}(0, 1).$$

This gives

$$\begin{aligned}\beta &= \text{Pr}\left(-c \le Z + \frac{n(p-p_0)}{\sqrt{npq}} \le c\right) \\ &= \text{Pr}\left(-c - \frac{\sqrt{n}(p-p_0)}{\sqrt{pq}} \le Z \le c - \frac{\sqrt{n}(p-p_0)}{\sqrt{pq}}\right) \\ &= \text{Pr}(-c-d \le Z \le c-d) \\ &= \Phi(c-d) - \Phi(-c-d),\end{aligned}$$

where

$$d \equiv \frac{\sqrt{n}(p-p_0)}{\sqrt{pq}}.$$

Notice that

$$-c-d \;=\; -z_{\alpha/2}\sqrt{\frac{p_0q_0}{pq}}-\frac{\sqrt{n}(p-p_0)}{\sqrt{pq}} \;\ll\; 0.$$

This implies, $\Phi(-c-d) \approx 0$, and so. . .

$$\beta \;\approx\; \Phi(c-d)$$

iff

$$c-d \;\approx\; \Phi^{-1}(\beta) \;=\; -z_\beta$$

iff

$$-z_\beta \;\approx\; z_{\alpha/2}\sqrt{\frac{p_0q_0}{pq}}-\frac{\sqrt{n}(p-p_0)}{\sqrt{pq}}.$$

After a little algebra, we finally(!) get

$$n \approx \left[\frac{z_{\alpha/2}\sqrt{p_0 q_0} + z_\beta\sqrt{pq}}{p - p_0}\right]^2.$$

Using similar reasoning, we can also get the sample size for the corresponding one-sided test:

$$n \approx \left[\frac{z_{\alpha}\sqrt{p_0 q_0} + z_\beta\sqrt{pq}}{p - p_0}\right]^2.$$