38. Normal Mean Tests (variance unknown)

Simple Hypothesis Test / Example

Two-Sample Pooled-t Hypothesis Test / Example

Two-Sample Approx-t Hypothesis Test / Example

Two-Sample Paired-t Hypothesis Test / Example

Simple Hypothesis Test

Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Nor(μ, σ^2), where σ^2 is *unknown*.

Two-sided test:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

We'll use $\bar{X}$ to estimate μ. If $\bar{X}$ is "significantly different" than μ_0, then we'll reject H_0. To determine what "significantly different" means, we'll also need to estimate σ^2.

Define

$$T_0 \equiv \frac{\bar{X} - \mu_0}{S/\sqrt{n}},$$

where S^2 is our old friend, the sample variance,

$$S^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \frac{\sigma^2\chi^2(n-1)}{n-1}.$$

If H_0 is true, then

$$T_0 = \frac{\frac{\bar{X}-\mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{S^2}{\sigma^2}}} \sim \frac{\text{Nor}(0,1)}{\sqrt{\frac{\chi^2(n-1)}{n-1}}} \sim t(n-1).$$

So

$$\text{Reject } H_0 \qquad \text{iff} \qquad |T_0| > t_{\alpha/2,n-1}.$$

One-Sided Tests:

$$H_0 : \mu \leq \mu_0 \quad \text{vs.} \quad H_1 : \mu > \mu_0$$

Reject H_0 iff $T_0 > t_{\alpha,n-1}$.

$$H_0 : \mu \geq \mu_0 \quad \text{vs.} \quad H_1 : \mu < \mu_0$$

Reject H_0 iff $T_0 < -t_{\alpha,n-1}$.

Example: Suppose we want to test at level 0.05 whether or not the mean of some process is 150.

We have $n = 15$, $\bar{X} = 152.18$, and $S^2 = 16.63$.

Then

$$T_0 \equiv \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = 2.07.$$

Since $t_{\alpha/2,n-1} = t_{.025,14} = 2.145$, we *fail to reject* H_0.

Two-Sample Normal Means Test when Variances are Unknown

Suppose we have the following set-up:

$$X_1, X_2, \ldots, X_{n_x} \overset{\text{iid}}{\sim} \text{Nor}(\mu_x, \sigma_x^2) \quad \text{and}$$

$$Y_1, Y_2, \ldots, Y_{n_y} \overset{\text{iid}}{\sim} \text{Nor}(\mu_y, \sigma_y^2),$$

where the samples are indep of each other, and σ_x^2 and σ_y^2 are *unknown*.

Which population has the larger mean?

Three Cases:

Pooled t-test: $\sigma_x^2 = \sigma_y^2 = \sigma^2$, say.

Approximate t-test: $\sigma_x^2 \neq \sigma_y^2$.

Paired t-test: (X_i, Y_i) observations paired.

Pooled t-Test

Suppose that $\sigma_x^2 = \sigma_y^2 = \sigma^2$ (unknown).

We'll look at the two-sided test to see if the means are different.

$$H_0 : \mu_x = \mu_y$$
$$H_1 : \mu_x \neq \mu_y$$

Sample means and variances from the two popns:

$$\bar{X} \equiv \frac{1}{n_x} \sum_{i=1}^{n_x} X_i \quad \text{and} \quad \bar{Y} \equiv \frac{1}{n_y} \sum_{i=1}^{n_y} Y_i$$

$$S_x^2 = \frac{\sum_{i=1}^{n_x}(X_i - \bar{X})^2}{n_x - 1} \quad \text{and} \quad S_y^2 = \frac{\sum_{i=1}^{n_y}(Y_i - \bar{Y})^2}{n_y - 1}.$$

Define the *pooled* variance estimator by

$$S_p^2 = \frac{(n_x - 1)S_x^2 + (n_y - 1)S_y^2}{n_x + n_y - 2}.$$

If H_0 is true, it can be shown that

$$S_p^2 \sim \frac{\sigma^2\chi^2(n_x+n_y-2)}{n_x+n_y-2},$$

and then the test statistic

$$T_0 \equiv \frac{\bar{X}-\bar{Y}}{S_p\sqrt{\frac{1}{n_x}+\frac{1}{n_y}}} \sim t(n_x+n_y-2).$$

Thus,

$$\text{Reject } H_0 \quad \text{iff} \quad |T_0| > t_{\alpha/2,n_x+n_y-2}.$$

One-Sided Tests:

$$H_0 : \mu_x \le \mu_y \quad \text{vs.} \quad H_1 : \mu_x > \mu_y$$

$$\text{Reject } H_0 \quad \text{iff} \quad T_0 \, > \, t_{\alpha,n_x+n_y-2}.$$

$$H_0 : \mu_x \ge \mu_y \quad \text{vs.} \quad H_1 : \mu_x < \mu_y$$

$$\text{Reject } H_0 \quad \text{iff} \quad T_0 \, < \, -t_{\alpha,n_x+n_y-2}.$$

Example: Catalyst X is currently used by a certain chemical process. If catalyst Y gives higher mean yield, we'll use it instead.

Thus, we want to test $H_0 : \mu_x \geq \mu_y$ vs. $H_1 : \mu_x < \mu_y$.

Suppose we have the following data:

$$n_x = 8, \ \bar{X} = 91.73, \ S_x^2 = 3.89$$

$$n_y = 8, \ \bar{Y} = 93.75, \ S_y^2 = 4.02$$

S_x^2 is pretty close to S_y^2, so we'll assume $\sigma_x^2 \approx \sigma_y^2$.

This justifies the use of the pooled variance estimator

$$S_p^2 = \frac{(n_x - 1)S_x^2 + (n_y - 1)S_y^2}{n_x + n_y - 2} = 3.955,$$

so that

$$T_0 = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}}} = -2.03.$$

Let's test at level $\alpha = 0.05$. Then

$$t_{\alpha,n_x+n_y-2} = t_{.05,14} = 1.761.$$

Since $T_0 < -t_{\alpha,n_x+n_y-2}$, we *reject* H_0.

Thus, we should probably use catalyst Y.

Approximate t-Test

Suppose that $\sigma_x^2 \neq \sigma_y^2$ (both unknown). As with our work with CI's, define

$$T_0^\star \equiv \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_x^2}{n_x} + \frac{S_y^2}{n_y}}} \approx t(\nu) \quad \text{(if } H_0 \text{ true)},$$

where the approximate degrees of freedom is given by

$$\nu \equiv \frac{\left(S_x^2/n_x + S_y^2/n_y\right)^2}{\frac{(S_x^2/n_x)^2}{n_x+1} + \frac{(S_y^2/n_y)^2}{n_y+1}} - 2.$$

9.38 Normal Mean Tests (var unknown)

2-sided	$H_0 : \mu_x = \mu_y$ $H_1 : \mu_x \neq \mu_y$	Reject H_0 iff $\vert T_0^\star\vert > t_{\alpha/2,\nu}$
1-sided	$H_0 : \mu_x \leq \mu_y$ $H_1 : \mu_x > \mu_y$	Reject H_0 iff $T_0^\star > t_{\alpha,\nu}$
1-sided	$H_0 : \mu_x \geq \mu_y$ $H_1 : \mu_x < \mu_y$	Reject H_0 iff $T_0^\star < -t_{\alpha,\nu}$

Example: Let's test $H_0 : \mu_x = \mu_y$ vs. $H_1 : \mu_x \neq \mu_y$ at level $\alpha = 0.10$.

Suppose we have the following data:

$$n_x = 15, \ \bar{X} = 24.2, \ S_x^2 = 10$$

$$n_y = 10, \ \bar{Y} = 23.9, \ S_y^2 = 20$$

S_x^2 isn't very close to S_y^2, so we'll assume $\sigma_x^2 \neq \sigma_y^2$.

Plug-and-chug to get

$$T_0^\star = 0.184,$$

$$\nu = 16.2 \rightarrow 16,$$

$$t_{\alpha/2,\nu} = t_{.05,16} = 1.746.$$

Since $|T_0^\star| < t_{\alpha/2,\nu}$, we *fail to reject* H_0.

(Actually, $T_0^\star$ was so small, we didn't need the tables.)

Paired t-Test

Again consider two competing normal pop'ns. Suppose we collect obs'ns from the two pop'ns in *pairs*.

The RV's within *different* pairs are *indep*. The two obs'ns within the *same* pair may *not* be indep — in fact, it's good for them to be positively correlated!

Example: One twin takes a new drug, the other takes a placebo.

In symbols,

$$\text{indep}\left\{\begin{array}{ll} \text{Pair } 1: & (X_1, Y_1) \\ \text{Pair } 2: & (X_2, Y_2) \\ \vdots & \vdots \\ \text{Pair } n: & \underbrace{(X_n, Y_n)}_{\text{not indep}} \end{array}\right.$$

Define the pair-wise differences,

$$D_i \equiv X_i - Y_i, \quad i = 1, 2, \ldots, n.$$

Note that $D_1, D_2, \ldots, D_n \overset{\text{iid}}{\sim} \text{Nor}(\mu_d, \sigma_d^2)$, where

$$\mu_d \;\equiv\; \mu_x - \mu_y \quad \text{and}$$

$$\sigma_d^2 \;\equiv\; \sigma_x^2 + \sigma_y^2 - 2\text{Cov}(X_i, Y_i).$$

Define the sample mean and var of the differences,

$$\bar{D} \;\equiv\; \sum_{i=1}^{n} D_i/n \;\sim\; \text{Nor}(\mu_d, \sigma_d^2/n)$$

$$S_d^2 \;\equiv\; \frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2 \;\sim\; \frac{\sigma_d^2 \chi^2(n-1)}{n-1}.$$

Then the test statistic is (assuming $\mu_d = 0$)

$$T_0 \equiv \frac{\bar{D}}{\sqrt{S_d^2/n}} \sim t(n-1).$$

Using the exact same manipulations as in the single-sample normal mean problem with unknown variance, we get the following. . .

9.38 Normal Mean Tests (var unknown)

2-sided	$H_0 : \mu_d = 0$ $H_1 : \mu_d \neq 0$	Reject H_0 iff $\lvert T_0 \rvert > t_{\alpha/2,n-1}$
1-sided	$H_0 : \mu_d \leq 0$ $H_1 : \mu_d > 0$	Reject H_0 iff $T_0 > t_{\alpha,n-1}$
1-sided	$H_0 : \mu_d \geq 0$ $H_1 : \mu_d < 0$	Reject H_0 iff $T_0 < -t_{\alpha,n-1}$

Recall $\mu_d = \mu_x - \mu_y$, so that, e.g., $\mu_d = 0$ iff $\mu_x = \mu_y$.

Example: Times for people to parallel park two cars.

Person	Park Honda	Park Cadillac	Difference
1	10	20	-10
2	25	40	-15
3	5	5	0
4	20	35	-15
5	15	20	-5

Let's test $H_0 : \mu_h = \mu_c$ at level $\alpha = 0.10$.

We see that $n = 5$, $\bar{D} = 9$, $S_d^2 = 42.5$.

This gives $T_0 = 3.087$.

Meanwhile, $t_{.05,4} = 2.13$, so we *reject* H_0.

Thus, we conclude that $\mu_h \neq \mu_c$ (and it's probably the case that Hondas are easier to park).