37. Normal Mean Tests (variance known)

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Simple Hypothesis Test

Suppose that $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Nor}(\mu, \sigma^2)$, where σ^2 is somehow *known* (not very realistic).

Two-sided test:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

We'll use $\bar{X}$ to estimate μ. If $\bar{X}$ is "significantly different" than μ_0, then we'll reject H_0. How much is "significantly different"?

Define

$$Z_0 \equiv \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}.$$

If H_0 is true, then $Z_0 \sim \text{Nor}(0,1)$, in which case

$$\Pr(-z_{\alpha/2} \le Z_0 \le z_{\alpha/2}) = 1 - \alpha.$$

A value of Z_0 outside the interval $[-z_{\alpha/2}, z_{\alpha/2}]$ is highly unlikely if H_0 is true. Therefore,

$$\text{Reject } H_0 \qquad \text{iff} \qquad |Z_0| > z_{\alpha/2}.$$

This assures us that

$$\Pr(\text{Type I error}) \;=\; \Pr(\text{Reject } H_0 \mid H_0 \text{ true}) \;=\; \alpha.$$

If $|Z_0| > z_{\alpha/2}$, then we're in the **rejection region**. (Also called **critical region**.)

If $|Z_0| \leq z_{\alpha/2}$, then we're in the **acceptance region**.

One-sided test:

$$H_0 : \mu \leq \mu_0$$

$$H_1 : \mu > \mu_0$$

A value of $Z_0 = (\bar{X} - \mu_0)/(\sigma/\sqrt{n})$ outside the interval $(-\infty, z_\alpha]$ is highly unlikely if H_0 is true. Therefore,

$$\text{Reject } H_0 \quad \text{iff} \quad Z_0 > z_\alpha.$$

If $Z_0 > z_\alpha$, this suggests $\mu > \mu_0$.

The other one-sided test:

$$H_0 : \mu \geq \mu_0$$

$$H_1 : \mu < \mu_0$$

A value of Z_0 outside the interval $[-z_\alpha, \infty)$ is highly unlikely if H_0 is true. Therefore,

$$\text{Reject } H_0 \qquad \text{iff} \qquad Z_0 < -z_\alpha.$$

If $Z_0 < -z_\alpha$, this suggests $\mu < \mu_0$.

Example: We examine the weights of 25 nine-year-old kids. Suppose we somehow know that the weights are normally distributed with $\sigma = 4$. The sample mean of the 25 weights is 42.

Test the hypothesis that the mean weight is 40. Keep the probability of Type I error = 0.05.

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu \neq \mu_0$$

Here we have

$$Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{42 - 40}{4/\sqrt{25}} = 2.5.$$

Since $|Z_0| = 2.5 > z_{\alpha/2} = z_{.025} = 1.96$, we *reject* H_0.

Notice that a lower α results in a higher $z_{\alpha/2}$. Then it's "harder" to reject H_0.

Example: If $\alpha = 0.01$, then $z_{.005} = 2.58$, and we *fail to reject* H_0 in the above example.

Definition: The p-**value** of a test is the smallest level of significance α that would lead to rejection of H_0.

For the two-sided normal mean test with known variance, we reject H_0 iff

$$|Z_0| > z_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$$

$$\text{iff} \quad \Phi(|Z_0|) > 1-\alpha/2$$

$$\text{iff} \quad \alpha > 2(1-\Phi(|Z_0|)).$$

Thus, for the two-sided test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

the p-value is $2(1 - \Phi(|Z_0|))$.

Similarly, for the one-sided test

$$H_0 : \mu \leq \mu_0$$
$$H_1 : \mu > \mu_0$$

$p = 1 - \Phi(Z_0)$.

Finally, for the one-sided test

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

we have $p = \Phi(Z_0)$.

Example: For the previous example,

$$p = 2(1 - \Phi(|Z_0|)) = 2(1 - \Phi(2.5)) = 0.0124.$$

Now we'll discuss how to design a test under constraints involving Type I and Type II errors. . . .

Test Design

Can we design a two-sided test such that

$$\Pr(\text{Type I error}) = \alpha \quad \text{and}$$

$$\Pr(\text{Type II error} \,|\, \mu = \mu_1 > \mu_0) = \beta?$$

I.e., can we design a two-sided test that will work when we require a Type I error bound α, and a Type II prob β (in the special case that the true mean μ happens to equal a user-specified value $\mu_1 > \mu_0$)?

Suppose the difference between the actual and hypothesized means is

$$\delta \equiv \mu - \mu_0 = \mu_1 - \mu_0.$$

(Without loss of generality, we'll assume $\mu_1 > \mu_0$.) Then it can be shown that the α and β design requirements can be achieved by taking a sample of size

$$n \approx \sigma^2 (z_{\alpha/2} + z_\beta)^2 / \delta^2.$$

Proof:

$$
\begin{aligned}
\beta &= \Pr(\text{Type II error} \,|\, \mu = \mu_1 > \mu_0) \\
&= \Pr(\text{Fail to Reject } H_0 \,|\, H_0 \text{ false } (\mu = \mu_1 > \mu_0)) \\
&= \Pr(|Z_0| \le z_{\alpha/2} \,|\, \mu = \mu_1) \\
&= \Pr(-z_{\alpha/2} \le Z_0 \le z_{\alpha/2} \,|\, \mu = \mu_1) \\
&= \Pr\left(-z_{\alpha/2} \le \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \le z_{\alpha/2} \,\middle|\, \mu = \mu_1\right) \\
&= \Pr\left(-z_{\alpha/2} \le \frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} \le z_{\alpha/2} \,\middle|\, \mu = \mu_1\right)
\end{aligned}
$$

Notice that

$$Z \equiv \frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} \sim \text{Nor}(0, 1).$$

This gives

$$\begin{aligned}
\beta &= \Pr\left(-z_{\alpha/2} \le Z + \frac{\sqrt{n}\delta}{\sigma} \le z_{\alpha/2}\right) \\
&= \Pr\left(-z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma} \le Z \le z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma}\right) \\
&= \Phi\left(z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma}\right)
\end{aligned}$$

Consider the second term in the previous expression, i.e.,

$$\Phi\left(-z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma}\right). \qquad (*)$$

First, notice that $-z_{\alpha/2} \ll 0$.

Second, since $\delta > 0$, we see that $\sqrt{n}\delta/\sigma < 0$.

These two facts imply that $(*) \approx 0$.

Thus, we only need to use the first term in the previous expression for β:

$$\beta \approx \Phi\Big(z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma}\Big)$$

iff

$$\Phi^{-1}(\beta) = -z_\beta \approx z_{\alpha/2} - \frac{\sqrt{n}\delta}{\sigma}$$

iff

$$\frac{\sqrt{n}\delta}{\sigma} \approx z_{\alpha/2} + z_\beta$$

iff

$$n \approx \sigma^2(z_{\alpha/2} + z_\beta)^2/\delta^2.$$

Recap: If you want to test $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$, and

(1) You know σ^2,

(2) You want Pr(Type I error) $= \alpha$, and

(3) You want Pr(Type II error) $= \beta$ if $\mu = \mu_1 (\neq \mu_0)$,

then you have to take $n \approx \sigma^2 (z_{\alpha/2} + z_\beta)^2 / \delta^2$ observations.

Similarly, if you're doing a *one-sided* test, it turns out that you need to take $n \approx \sigma^2(z_\alpha + z_\beta)^2/\delta^2$ obsns.

Example: Weights of 9-yr-old kids are normal with $\sigma = 4$. How many obsns should we take if we wish to test $H_0 : \mu = 40$ vs. $H_1 : \mu \neq 40$, and we want $\alpha = 0.05$, and $\beta = 0.05$ if μ happens to actually equal $\mu_1 = 42$?

$$n \approx \frac{\sigma^2}{\delta^2}(z_{\alpha/2} + z_\beta)^2 = \frac{16}{4}(1.96 + 1.645)^2 = 51.98.$$

In other words, you need about 52 observations.

Two-Sample Normal Means Test when Variances are Known

Suppose we have the following set-up:

$$X_1, X_2, \ldots, X_{n_x} \overset{\text{iid}}{\sim} \text{Nor}(\mu_x, \sigma_x^2) \quad \text{and}$$

$$Y_1, Y_2, \ldots, Y_{n_y} \overset{\text{iid}}{\sim} \text{Nor}(\mu_y, \sigma_y^2),$$

where the samples are indep of each other, and σ_x^2 and σ_y^2 are somehow *known*.

Which population has the larger mean?

Here's the two-sided test to see if the means are different.

$$H_0 : \mu_x = \mu_y$$

$$H_1 : \mu_x \neq \mu_y$$

Define the test statistic

$$Z_0 = \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}}.$$

If H_0 is true (i.e., the means are equal), then

$$Z_0 = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \sim \text{Nor}(0, 1).$$

Thus, as before,

$$\text{Reject } H_0 \quad \text{iff} \quad |Z_0| > z_{\alpha/2}.$$

Using more of the same reasoning as before, we get the following one-sided tests.

$$H_0 : \mu_x \le \mu_y \quad \text{vs.} \quad H_1 : \mu_x > \mu_y$$

Reject H_0 iff $Z_0 = \frac{\bar{X}-\bar{Y}}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}} > z_\alpha$.

$$H_0 : \mu_x \ge \mu_y \quad \text{vs.} \quad H_1 : \mu_x < \mu_y$$

Reject H_0 iff $Z_0 = \frac{\bar{X}-\bar{Y}}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}} < -z_\alpha$.

Example: Suppose we want to test $H_0 : \mu_x = \mu_y$ vs. $H_1 : \mu_x \neq \mu_y$, and we have the following data:

$$n_x = 10,\ \bar{X} = 824.9,\ \sigma_x^2 = 40$$
$$n_y = 10,\ \bar{Y} = 818.6,\ \sigma_y^2 = 50$$

Then

$$Z_0 = \frac{824.9 - 818.6}{\sqrt{\frac{40}{10} + \frac{50}{10}}} = 2.10.$$

If $\alpha = 0.05$, then $|Z_0| = 2.10 > z_{\alpha/2} = 1.96$, and so we *reject* H_0.