Continuous Random Variables — Modules

23. Starters — Uniform, Exponential, and Related Distrns

24. Others (except for the Normal Distrn)

Uniform, Exponential, and Related Distributions

Uniform

Exponential

Erlang

Gamma

Uniform(a, b) Distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Previous work showed that

$$\mathsf{E}[X] = \frac{a+b}{2} \qquad \mathsf{Var}(X) = \frac{(a-b)^2}{12}.$$

We can also derive the m.g.f.,

$$M_X(t) = \mathsf{E}[e^{tX}] = \int_a^b e^{tx} \frac{1}{b-a} \, dx = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

Exponential(λ) Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Previous work showed that the c.d.f. $F(x) = 1 - e^{-\lambda x}$,

$$\mathsf{E}[X] = 1/\lambda, \text{ and } \mathsf{Var}(X) = 1/\lambda^2.$$

We also derived the m.g.f.,

$$M_X(t) = \mathsf{E}[e^{tX}] = \int_0^\infty e^{tx} f(x)\, dx = \frac{\lambda}{\lambda - t}, \quad t < \lambda.$$

Memoryless Property of Exponential

Theorem: Suppose that $X \sim \text{Exp}(\lambda)$. Then for positive s, t, we have

$$\Pr(X > s + t | X > s) = \Pr(X > t).$$

Similar to the discrete Geometric distribution, the prob that X will survive an additional t time units is the (unconditional) prob that it will survive at least t — it forgot that it made it past time s!

Proof:

$$\Pr(X > s + t | X > s)$$

$$= \frac{\Pr(X > s + t \cap X > s)}{\Pr(X > s)}$$

$$= \frac{\Pr(X > s + t)}{\Pr(X > s)} \quad (t \text{ positive})$$

$$= \frac{1 - F(s + t)}{1 - F(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda(s)}}$$

$$= e^{-\lambda(t)} = \Pr(X > t).$$

Example: Suppose that the life of a lightbulb is exponential with a mean of 1000 hours. If the light survives 1000 hours, what's the prob that it'll survive another 1000?

$$\begin{aligned}
\Pr(X > 2000 | X > 1000) &= \Pr(X > 1000) \\
&= e^{-\lambda x} \\
&= e^{-(1/1000)(1000)} \\
&= e^{-1} = 0.370.
\end{aligned}$$

Remark: The exponential is the *only* cts distrn with the memoryless property.

Remark: Look at E$[X]$ and Var(X) for the Geometric distrn and see how they're similar to those for the exponential. (Not a coincidence.)

The Exponential is also related to the Poisson!

Let X be the amount of time until the first arrival in a Poisson process with rate λ. Then $X \sim \text{Exp}(\lambda)$.

Proof: Note that the number of arrls in $[0, x]$ is Pois(λx).

$$\begin{aligned} F(x) &= \Pr(X \leq x) = 1 - \Pr(\text{no arrls in } [0, x]) \\ &= 1 - \frac{e^{-\lambda x}(\lambda x)^0}{0!} \quad \text{Pois}(\lambda x) \\ &= 1 - e^{-\lambda x}. \end{aligned}$$

Erlang Distribution

Definition: Suppose $X_1, \ldots, X_k \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$, and let $S = \sum_{i=1}^{k} X_i$. Then S has the **Erlang$_k$ ditribution** with parameter λ.

The Erlang is simply the sum of i.i.d. exponentials.

Special Case: $\text{Erlang}_1(\lambda) \sim \text{Exp}(\lambda)$.

Erlang Properties

The p.d.f. and c.d.f. are

$$f(s) = \frac{\lambda^k e^{-\lambda s} s^{k-1}}{(k-1)!}, \quad s \geq 0,$$

$$F(s) = 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda s}(\lambda s)^i}{i!}.$$

Notice that the c.d.f. is the sum of a bunch of Poisson probabilities. (Won't do it here, but this observation helps in the derivation of the c.d.f.)

Expected Value, Variance, and m.g.f.:

$$\mathsf{E}[S] = \mathsf{E}[\sum_{i=1}^{k} X_i] = \sum_{i=1}^{k} \mathsf{E}[X_i] = k/\lambda$$

$$\mathsf{Var}(S) = k/\lambda^2$$

$$M_S(t) = \left(\frac{\lambda}{\lambda - t}\right)^k.$$

Example: Suppose X and Y are i.i.d. Exp(2). Find $\Pr(X + Y < 1)$.

$$\Pr(X + Y < 1) = 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda s}(\lambda s)^i}{i!}$$

$$= 1 - \sum_{i=0}^{2-1} \frac{e^{-(2 \cdot 1)}(2 \cdot 1)^i}{i!}$$

$$= 0.594$$

Gamma Distribution

Definition: X has the **gamma distribution** with parameters $\alpha > 0$ and $\lambda > 0$ is it has p.d.f.

$$f(x) \;=\; \frac{\lambda^{\alpha} e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)}, \;\; x \geq 0,$$

where

$$\Gamma(\alpha) \;\equiv\; \int_0^{\infty} t^{\alpha-1} e^{-6}\, dt$$

is the *gamma function*.

Remark: The gamma distrn generalizes the Erlang distrn (where α has to to be an integer).

Remark: If α is a positive integer, then $\Gamma(\alpha) = (\alpha - 1)!$.

Party trick: $\Gamma(1/2) = \sqrt{\pi}$.