3.21 Geometric, Negative Binomial Distrns

Geometric Distribution

Memoryless Property of Geometric

Negative Binomial Distribution

Comparison of Binomial and Negative Bin

Geometric(p) Distribution

Suppose we consider an infinite sequence of indep Bern(p) trials.

Let Z equal the number of trials *until the first success* is obtained. The event $Z = k$ corresponds to $k - 1$ failures, and then a success. Thus,

$$\Pr(Z = k) \;=\; q^{k-1}p, \quad k = 1, 2, \ldots .$$

Z has the **Geometric(p) distribution**.

The mgf of the Geom(p) is

$$\begin{aligned} M_Z(t) &= \mathsf{E}[e^{tZ}] = \sum_{k=1}^{\infty} e^{tk} q^{k-1} p \\ &= pe^t \sum_{k=0}^{\infty} (qe^t)^k \\ &= \frac{pe^t}{1-qe^t}, \quad \text{for } qe^t < 1. \end{aligned}$$

So

$$M_Z(t) = \frac{pe^t}{1-qe^t}, \quad \text{for } t < \ell\text{n}(1/q).$$

Thus,

$$
\begin{aligned}
\mathsf{E}[Z] &= \frac{d}{dt}M_Z(t)\bigg|_{t=0} \\
&= \frac{(1-qe^t)(pe^t)-(-qe^t)(pe^t)}{(1-qe^t)^2}\bigg|_{t=0} \\
&= \frac{pe^t}{(1-qe^t)^2}\bigg|_{t=0} \\
&= \frac{p}{(1-q)^2} = \frac{1}{p}.
\end{aligned}
$$

Similarly, after a lot of algebra,

$$\mathsf{E}[Z^2] \;=\; \frac{d^2}{dt^2} M_Z(t)\Big|_{t=0} \;=\; \frac{2-p}{p^2},$$

so that

$$\mathsf{Var}(Z) \;=\; \mathsf{E}[Z^2] - (\mathsf{E}[Z])^2 \;=\; \frac{2-p}{p^2} - \frac{1}{p^2} \;=\; \frac{q}{p^2}.$$

Example: Toss a die repeatedly. What's the prob that we observe a '3' for the first time on the 8th toss?

Answer: The number of tosses we need is $Z \sim \text{Geom}(1/6)$. $\Pr(Z = 8) = (5/6)^7(1/6)$.

How many tosses would we expect to take?

Answer: $\text{E}[Z] = 1/p = 6$ tosses.

Memoryless Property of Geometric

Theorem: Suppose that $Z \sim \text{Geom}(p)$. Then for positive integers s, t, we have

$$\Pr(Z > s + t | Z > s) \; = \; \Pr(Z > t).$$

Why is it the **memoryless property**? Well, Tommy, if an event hasn't occurred by time s, the prob that it will occur after an additional t time units is the same as the (unconditional) prob that it will occur after time t — it forgot that it made it past time s!

Proof:

$$\text{Pr}(Z > s + t | Z > s)$$

$$= \frac{\text{Pr}(Z > s + t \cap Z > s)}{\text{Pr}(Z > s)}$$

$$= \frac{\text{Pr}(Z > s + t)}{\text{Pr}(Z > s)} \quad (t \text{ positive})$$

$$= \frac{\sum_{j=s+t+1}^{\infty} q^{j-1}p}{\sum_{j=s+1}^{\infty} q^{j-1}p} = \frac{q^{s+t}\sum_{j=0}^{\infty} q^j}{q^s \sum_{j=0}^{\infty} q^j}$$

$$= q^t.$$

Meanwhile,

$$\begin{aligned}
\Pr(Z > t) &= \sum_{j=t+1}^{\infty} q^{j-1} p \\
&= pq^t \sum_{j=0}^{\infty} q^j \\
&= \frac{pq^t}{1-q} \\
&= q^t.
\end{aligned}$$

Thus, $\Pr(Z > s + t | Z > s) = \Pr(Z > t)$. Done.

Fun Fact: The Geom(p) is the only discrete distribution with the memoryless property.

Not-so-Fun Fact: Some books define the Geom(p) as the number of Bern(p) *failures* until you observe a success. $\#$ failures $= \#$ trials $- 1$. You should be aware of this inconsistency, but don't worry about it for now.

Negative Binomial Distribution (aka Pascal distrn)

Suppose we consider an infinite sequence of indep Bern(p) trials.

Now let Z equal the number of trials *until the* r*th success* is obtained. $Z = r, r+1, \ldots$. The event $Z = k$ corresponds to exactly $r-1$ successes by time $k-1$, and then the rth success at time k.

‘FFFFFSFS’ corresponds to $Z = 7$ trials until the $r =$ 2nd success.

Notation: $Z \sim$ NegBin(r, p).

Remark: As with the Geom(p), the exact definition of the NegBin depends on what book you’re reading.

Theorem: If $Z_1, \ldots, Z_r \overset{\text{iid}}{\sim}$ Geom(p), then $Z = \sum_{i=1}^{n} Z_i \sim$ NegBin(r, p).

Proof: Won't do it here, but you can use the mgf technique.

Anyhow, it makes sense if you think of Z_i as the number of trials after the $(i-1)$st success up to and including the ith success.

Since the Z_i's are i.i.d., the above theorem gives:

$$\mathsf{E}[Z] = r\mathsf{E}[Z_i] = r/p,$$

$$\mathsf{Var}(Z) = r\mathsf{Var}(Z_i) = rq/p^2,$$

$$M_Z(t) = [M_{Z_i}(t)]^r = \left(\frac{pe^t}{1 - qe^t}\right)^r.$$

Just to be complete, let's get the pmf of Z.

$Z = k$ iff get exactly $r-1$ successes by time $k-1$, and then the rth success at time k. So...

$$\begin{aligned}\Pr(Z = k) &= \left[\binom{k-1}{r-1} p^{r-1} q^{k-r}\right] p, \quad k = r, r+1, \ldots \\ &= \binom{k-1}{r-1} p^{r} q^{k-r}, \quad k = r, r+1, \ldots.\end{aligned}$$

How are the Bin and NegBin Related?

$X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bern}(p) \;\Rightarrow\; X \equiv \sum_{i=1}^{n} X_i \sim \text{Bin}(n,p).$

$Z_1, \ldots, Z_r \overset{\text{iid}}{\sim} \text{Geom}(p) \;\Rightarrow\; Z \equiv \sum_{i=1}^{r} Z_i \sim \text{NegBin}(r,p).$

$\text{E}[X] = np$, $\text{Var}(X) = npq$.

$\text{E}[Z] = r/p$, $\text{Var}(Z) = rq/p^2$.